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SOME REMARKS ON THREE-VALUED IMPLICATIVE SENTENTIAL CALCULI

Let $\rightarrow, \vee, \wedge, \neg, f$ be functions defined on the set $\{2, 1, 0\}$ for which the following conditions are satisfied:

(i) (A) $(x \rightarrow x) = 2$
(B) if $(x \rightarrow y) = 2$ and $x = 2$, then $y = 2$
(C) $(x \rightarrow 2) = 2$
(D) if $(x \rightarrow y) = 2$ and $(y \rightarrow z) = 2$, then $(x \rightarrow z) = 2$
(E) if $(x \rightarrow y) = 2$ and $(y \rightarrow x) = 2$, then $x = y$

(ii) $x \wedge y = max(x, y)$
(iii) $x \vee y = min(x, y)$
(iv) $fx = 0$
(v) $\neg x = (x \rightarrow fx)$

The matrix

$$Z = \langle \{2, 1, 0\}, \{2\}, \rightarrow, \vee, \wedge, \neg, f \rangle$$

will be called an *implicative three-valued matrix*.

Remark 1. The conditions (A) - (B) are analogous to those given by Rasiowa for the operation $\rightarrow$ in an implicative algebra (see [1]).

Remark 2. There exist 32 mutually different three-valued matrices

$$Z_{(k)} = \langle \{2, 1, 0\}, \{2\}, \rightarrow, \vee, \wedge, \neg_k, f \rangle$$

the functions of which satisfy the conditions (i) - (v).

By a (k)-*implicative three-valued sentential calculus* we shall understand any couple $S_k = (L_k, Cn_{(k)})$ where $L_k = \langle \mathcal{L}_k, \rightarrow_k, \vee, \wedge, \neg_k, f \rangle$ is a sentential language and $Cn_{(k)}$ is a consequence determined by the (k)-implicative three-valued matrix $Z_{(k)}$.

From the remark given above it follows that there exists 32 such three-valued calculi. The individual calculi S_k differ from one another in the connectives of implication $\to_k$ and negation $\neg_k$, while the connectives of disjunction, conjunction and falsum are determined by the same tables. The aim of this paper is to compare these calculi with respect to mutual definability of the connectives $\to_k$ and $\neg_k$.

Let $S_k = (L_k, Cn_{(k)})$ be a (k)-implicative three-valued sentential calculus. We shall say that S_k is *definitionally complete* if and only if the (k)-implicative matrix $Z_{(k)}$ is functionally complete.

Let $S_k = (L_k, Cn_{(k)})$, $S_i = (L_i, Cn_{(i)})$ be two sentential calculi. S_k will be said to be *definable* in S_i, in symbols

$$S_k \subset\!\cdot\; S_i$$

if and only if the connective of implication $\to_k$ (and therefore that of $\neg_k$) is definable in the calculus S_i. If $S_k \subset\!\cdot\; S_i$ and $S_i \subset\!\cdot\; S_k$ we shall say that S_k and S_i are *mutually reconstructable* in symbols

$$S_k \sim S_i$$

The symbols $S_p, S_Ł, S_H, S_1$ will be used to denote definitionally complete calculus, three-valued logic of Łukasiewicz, Heyting's three-valued calculus and the calculus with an implication $\to_1$ and negation $\neg_1$ being determined by the table

$\to_1$	2	1	0	$\neg_1$
2	2	1	1	1
1	2	2	2	2
0	2	1	2	2

respectively.

Now we define the sets $\overline{S}_p, \overline{S}_Ł, \overline{S}_H, \overline{S}_1$ as follows:

$$\begin{aligned}
\overline{S}_p &= \{S_k : S_k \sim S_p\}\\
\overline{S}_Ł &= \{S_k : S_k \sim S_Ł\}\\
\overline{S}_H &= \{S_k : S_k \sim S_H\}\\
\overline{S}_1 &= \{S_k : S_k \sim S_1\}
\end{aligned}$$

The following theorem holds:

THEOREM 2. *The mutual reconstructability relation $\sim$ determines a decomposition of the set of 32 calculi into four equivalence classes of $\sim$, namely $\overline{S}_p$, $\overline{S}_Ł$, $\overline{S}_H$, $\overline{S}_1$, such that two calculi S_i, S_j belong to the same equivalence class if and only if $S_i \sim S_j$.*

Let $\overline{S}_k$, $\overline{S}_i$ be two equivalence classes. We shall say that $\overline{S}_k$ is definable in $\overline{S}_i$, in symbols

$$\overline{S}_k \subset\!\cdot\; \overline{S}_i$$

if and only if there exist two calculi $S_j \in \overline{S}_k$ and $S_1 \in \overline{S}_i$ such that $S_j \subset\!\cdot\; S_1$.

The following theorem gives relations between $\overline{S}_p$, $\overline{S}_Ł$, $\overline{S}_H$, $\overline{S}_1$.

THEOREM 3.

(i) $\overline{S}_Ł \subset\!\cdot\; \overline{S}_p$, $\overline{S}_H \subset\!\cdot\; \overline{S}_p$, $\overline{S}_1 \subset\!\cdot\; \overline{S}_p$,
(ii) $\overline{S}_p \not\subset\!\cdot\; \overline{S}_Ł$, $\overline{S}_p \not\subset\!\cdot\; \overline{S}_H$, $\overline{S}_p \not\subset\!\cdot\; \overline{S}_1$,
(iii) $\overline{S}_Ł \not\subset\!\cdot\; \overline{S}_1$, $\overline{S}_1 \not\subset\!\cdot\; \overline{S}_Ł$,
(iv) $\overline{S}_H \subset\!\cdot\; \overline{S}_Ł$, $\overline{S}_Ł \not\subset\!\cdot\; \overline{S}_H$,
(v) $\overline{S}_H \not\subset\!\cdot\; \overline{S}_1$, $\overline{S}_1 \not\subset\!\cdot\; \overline{S}_H$.

References

[1] H. Rasiowa, **An algebraic approach to non classical logics**, North-Holland Publ. Co. Amsterdam, PWN, Warszawa, 1974.

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