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THE MODAL SYSTEM S_3 AND SCI

Let FM be the set of all formulas built of sentential variables $(p, q, r, \ldots)$ by means of usual truth-functional connectives $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$ and, in addition, the unary connective $\Box$ and the binary identity connective $\equiv$. The set of all truth-functional tautologies in FM is denoted by TFT. The set $S \subseteq FM$ is called a *modal system* iff $TFT \subseteq S$ and S is closed under the following rules.

(MP) The rule of modus ponens $\alpha, \alpha \Rightarrow \beta / \beta$.
(SB) The rule of substitution of formulas for sentential variables.
(RF) The rule of mutual replacement of $\alpha \equiv \beta$ by $\Box(\alpha \Leftrightarrow \beta)$.

The formula $\gamma[s/\alpha]$ is the result of substituting the formula α for the variable s in the formula γ.

According to [1], the system S_3 is the smallest modal system containing all the following formulas:

(0) $\Box\alpha$ whenever α is in TFT
(1) $\Box p \Rightarrow p$
(2) $\Box(\Box p \Rightarrow p)$
(3) $\Box(\Box(p \Rightarrow q) \Rightarrow \Box(\Box p \Rightarrow \Box q))$.

One may easily show that the system S_3 contains also the following formulas:

(4) $\alpha \equiv \beta$ whenever $\alpha \Leftrightarrow \beta$ is in TFT.
(5) $\Box(p \Rightarrow q) \Rightarrow \Box(\Box p \Rightarrow \Box q)$
(6) $\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$

(7) $\Box(p \wedge q) \Rightarrow (\Box p \wedge \Box q)$
(8) $(p \equiv q) \Rightarrow (\Box p \Leftrightarrow \Box q)$
(9) $(p \equiv q) \Rightarrow p \Leftrightarrow q$
(10) $(p \equiv q) \Rightarrow (\neg p \equiv \neg q)$
(11) $(p \equiv q) \wedge (r \equiv s) = [(p * r) \equiv (q * s)]$

The asterisk $*$ represents here any binary truth-functional connective.

(12) $p \equiv p$
(13) $(p \equiv q) \equiv \Box(p \Leftrightarrow q)$
(14) $(p \equiv q) \Rightarrow (\Box p \equiv \Box q)$
(15) $(p \equiv q) \wedge (r \equiv s) \Rightarrow [(p \equiv r) \equiv (q \equiv s)]$
(16) $(\alpha \equiv \beta) \Rightarrow (\gamma[s/\alpha] \equiv \gamma[s/\beta])$
(17) $\Box p \equiv (p \equiv \alpha)$ for any α in TFT.

By (12), (9), (10), (11), (15) we infer that S_3 is an invariant theory in SCI. Here, we will apply that terminology and notation of [2]. Thus, by (4) and (17), S_3 is a Boolean theory. Furthermore, by (13), S_3 contains WB_0.

If D_1 is the set A_0 supplemented with the formulas (2) and (3), then clearly

$$S_3 = Cn(D_1)$$

On the other hand, it can be shown that

$$S_3 = Cn(D_2)$$

where D_2 is the set A_0 supplemented with (2) and two following formulas:

$$\Box(p \wedge q) \equiv (\Box p \wedge \Box q)$$

$$\Box(p \equiv q) \Rightarrow (\Box p \equiv \Box q))$$

It is known that $\Box(p \equiv p)$ is not in S_3. This can be easily shown by considering an appropriate Boolean SCI-model. It follows that the modal system S_3 does not occur in the sequence of Boolean theories investigated in [2].

References

[1] E. J. Lemmon, *New foundations for Lewis modal systems*, **The Journal of Symbolic Logic** 22, no. 2 (1957), pp. 176–186.

[2] R. Wawrzyńczak, *Some Boolean theories in SCI*, this **Bulletin**, vol. 2 (1973), no. 3, pp. 197–204.

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