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CLASSICALLY AXIOMATIZABLE MODAL PROPOSITIONAL CALCULI CONTAINING THE SYSTEM T OF FEYS – VON WRIGHT*

This is an abstract of the paper presented at the seminar of the Section of Logic, Institute of Mathematics Nicholas Copernicus University, held by Professor Jerzy Kotas, Toruń, November 1975. The complete version of the paper will be published in Studia Logica.

The aim of this paper is to give some criterion of the classical axiomatizability of normal modal propositional calculi containing the system T of Feys-von Wright. Making use of the criterion we partly solve the problem formulated by Lemmon in [2] and prove that modal propositional calculi $T^*_k (k \geqslant 1)$ defined in [1] are not classically axiomatizable.

We use the well-known logical and set-theoretical notation. The symbol ω denotes the set of natural numbers, elements of which are represented by $i, j, k, m, n, \ldots$. Logical connectives of negation, material implication, necessity and possibility will be denoted N, C, L, M, respectively.

The propositional calculi in the paper will be treated as pairs $\langle R, A\rangle$, where R is the set of primitive rules and A – the set of axioms of the given propositional calculus. For any $\langle R, A\rangle$, by $Perm(R, A)$ will be denoted the set of all permissible rules in it, and by $Cn(R, A)$ will be denoted the least set of formulas containing A and closed on the rules R. We put $L^0\alpha = \alpha$ and $L^{k+1}\alpha = LL^k\alpha$. The abbreviation $M^k\alpha$ is defined in a similar way.

In further consideration we shall use the following deduction rules:

(r_1) : substitution rule,
(r_2) : if α and $C\alpha\beta$, then β,
(r^n_3) : if $L^n\alpha$, then $LL^n\alpha$.

*As abstract this article is not to be reviewed.

We see that (r_3^0) denotes the Godel's rule. We put $R_N = \{r_1, r_2, r_3^0\}$ and $L^n A = \{L^n\alpha : \alpha \in A\}$ for any set A of formulas.

Our discussion is restricted to the class of modal propositional calculi containing the system T of Feys-von Wright and such that set R_N constitutes the set of primitive rules, while the set of axioms is finite. The class will be denoted by $\underline{K}$. The modal propositional calculus $\langle R_N, A\rangle$ is classically axiomatizable (symbolically $\langle R_N, A\rangle \in AKS$) iff there exists the finite set B of formulas such that $Cn(\{r_1, r_2\}, B) = Cn(R_N, A)$.

Theorem 1. *For every $\langle R_N, A\rangle \in \underline{K}$ the following conditions are equivalent*

(I) $\exists n \in \omega : (r_3^n) \in Perm(\{r_1, r_2\}, L^n(A \cup \{CLCpqCLpLq, CLpp\}) \cup \{CLpp\}$
(II) $\langle R_N, A\rangle \in AKS$.

As it can easily be seen, the Theorem 1 provides certain criterion which helps to confirm whether given modal propositional calculus is or not classically axiomatizable. It appears that with the help of the criterion one may settle that class of modal propositional calculi classically axiomatizable containing the system T is very large. Namely, the following theorem results from the previous one.

Theorem 2. *For every $\langle R_N, A\rangle \in \underline{K}$ holds: if there exists $n \in \omega$, such that $CL^n pL^{n+1}p \in Cn(R_N, A)$, then $\langle R_N, A\rangle \in AKS$.*

Now, on the base of the above remarks it might be stated the following:

Corollary. *For every $\langle R_N, A\rangle \in \underline{K}$ holds: if $\langle R_N, A\rangle$ has finitely many modalities, then $\langle R_N, A\rangle \in AKS$.*

Note that the above corollary gives a partial (positive) solution of the problem stated by Lemmon in [2].

At present, on the base the Theorem 1, we shall show that modal propositional calculi $T_k^*(k \geqslant 1)$ discussed in [1] are not classically axiomatizable. Following our convention (that is treating propositional calculi as pairs) modal propositional calculi $T_k^*(k \geqslant 1)$ are given by $\langle R_N, A_k\rangle$, where A_k constitute the set of following formulas:

$CCpqCCqrCpr, CpCNpq, CCNppp, CLpp,$
$CLCpqCLpLq, CL^k pM^k L^{k+1}p.$

Consider the family of matrices

$\underline{U} = \langle \underline{B}(X_n) : D_n, \neg, \cup, \cap, C_n \rangle, n \in \omega$

The universum $\underline{B}(X_n)$ of the $\underline{U}_n$ is the class of all subsets of the set $X_n = \{1, 2, \ldots, n+1, n+2\}$.

The set of distinguished values is given as

$$D_n = \{B : B \in \underline{B}(X_n) \text{ and } n+1 \in B\}$$

The symbols $\cup, \cap, \neg$ denote the union, intersection and complementation operations, respectively. For the definition of $C_n : \underline{B}(X_n) \to \underline{B}(X_n)$ we define a relation R_n as follows:

$$R_n \subseteq X_n \times X_n$$
$$R_n = \{\langle i, j \rangle : ((i \neq 1) \wedge (i = j)) \vee ((i = j+1) \wedge (j < n+1)) \vee \vee((j > n+1) \wedge (i, j - \text{arbitrary}))\}$$

For any set $B \in \underline{B}(X_n)$

$C_n = \{i : \langle i, j \rangle \in R_n \text{ and } j \in B\} \cup \{1\}$

The interior X_n is defined as usual. The unit of the matrix $\underline{U}_n$ is the set X_n.

Let h be any homomorphism from the algebra of formulas into the algebra of matrices $\underline{U}_n$, and $k \geqslant 1$. It is easy to check, that for any formula $\alpha \in A_k$ hold:

(1) $h(\alpha) = X_n$
(2) $h(L^n \alpha) = \{n+1, n+2\}$
(3) $h(L^{n+1} \alpha) = \{n+2\}$.

From (3) it follows that $h(L^{n+1}\alpha)$ does not belong to the filter D_n. Since, $L^{n+1}\alpha \notin E(\underline{U}_n)$, where $E(\underline{U}_n)$ is the set of all formulas true in U_n. Therefore

$(r_3^n) \notin Perm(\{r_1, r_2\}, L^n A_k \cup \{CLpp\})$.

This, by arbitratiness of $n \in \omega$ and Theorem 1, allow us to note:

Theorem 3. $\langle R_N, A_k \rangle \notin AKS$, $(k \geqslant 1)$.

References

[1] J. J. Błaszczuk, W. Dziobiak, *Modal systems related to* $S4_n$ *of Sobociński*, forthcoming in **Studia Logica**.

[2] E. J. Lemmon, *Some results on finite axiomatizability in modal logic*, **Notre Dame Journal of Formal Logic**, vol. VI (1965), pp. 301–308.

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