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CHARACTERIZATION OF FINITELY AXIOMATIZABLE SETS ON THE BASIS OF A SYSTEM OF THE PROPOSITIONAL CALCULUS*

The complete text of this paper will appear in Functiones et Approximatic vol. IV.

The empty set and the set of all sentential variables are denoted by $\emptyset$ and At, respectively. The functions mapping the set At into the set $\{0, 1\}$ are called valuations. We designate by $v|Z$ the function obtained from the valuation v by restricting it to the set of variables Z. And analogously we designate by $K|Z$ the set of functions obtained from the valuations $v \in K$ by restricting them to the set Z.

Our considerations are devoted to the classical axiomatic system of the propositional calculus with the implication and negation connectives and the rule of detachment (Rasiowa [2]).

A set formulas X is said to be finitely axiomatizable iff there exists a finite set of formulas Y such that the set of all those formulas which can be inferred from axioms of the propositional calculus and from the set of formulas Y by means of the detachment rule is equal to the set X.

Let K be an arbitrary set of valuations. A set of formulas X is called K-saturated (cf. Pogorzelski [1], the set Sat_e) iff X is the set of all those formulas, which have the value 1 for all valuations belonging to the set K.

Definition. We say that a set of valuations K satisfies the condition W iff there exists sets of variables Z_1, Z_2 such that

$Z_1 \cup Z_2 = At$,
$Z_1 \cap Z_2 = \emptyset$,

*As abstract this article is not to be reviewed.

the set Z_1 is finite,
$K|Z_2 = \{0,1\}^{Z_2}$,

if $Z_1 \neq \emptyset$ then for every function $v \in K|Z_1$ and for every function $w \in K|Z_2$ there exists a valuation $u \in K$ such that

$$v = u|Z_1 \text{ and } w = u|Z_2.$$

Theorem. *A consistent set of formulas X is finitely axiomatizable iff there exists a set of valuations K fulfilling the condition W such that the set X is K-saturated.*

The proof of above Theorem yields a following simple procedure for finding axiom systems for some sets of formulas.

Let X be a K-saturated set of formulas and let the set of valuations K fulfills the condition W.

If $K = \{0,1\}^{At}$ then, of course, the empty set is the axiom system of the set X. Otherwise $Z_1 = \emptyset$. We denote then by Z_3 an arbitrary set of these variables $q \in Z_1$, which for arbitrary valuations $v, u \in K$ fulfill the condition $v(q) = u(q)$.

If the set $Z_1 - Z_3$ is non-empty, then we take the following notation:

$$\begin{aligned} Z_1 - Z_3 &= \{p_1, p_2, \ldots, p_n\}, \\ K(Z_1 - Z_3) &= \{v_1, v_2, \ldots, v_s\}, \\ p_i^1 &= p_i, \\ p_i^0 &= \sim p_i; \text{ for } i = 1, 2, \ldots, n, \end{aligned}$$

$$A_j = \begin{cases} p_1^{v_j(p_1)} \Rightarrow (p_2^{v_j(p_2)} \Rightarrow \ldots \Rightarrow (p_{n-1}^{v_j(p_{n-1})} \Rightarrow \sim p_n^{v_j(p_n)}) \ldots) \text{ if } n > 1 \\ \sim p_1^{v_j(p_1)} \text{ if } n = 1; \text{ for } j = 1, 2, \ldots, s, \end{cases}$$

$$A = \begin{cases} A_1 \Rightarrow (A_2 \Rightarrow \ldots \Rightarrow (A_{s-1} \Rightarrow \sim A_s) \ldots) & \text{if } s > 1, \\ \sim A_1 & \text{if } s = 1. \end{cases}$$

If the set Z_3 is non-empty, then we take the following notations:

$$\begin{aligned} Z_3 &= \{q_1, q_2, \ldots, q_m\}, \\ K|Z_3 &= \{w\}. \end{aligned}$$

A set Y defined as follows:

$$Y = \begin{cases} \{A, q_1^{w(q_1)}, q_2^{w(q_2)}, \ldots, q_m^{w(q_m)}\} \text{ if } Z_1 - Z_3 \neq \emptyset \text{ and } Z_3 \neq \emptyset, \\ \{A\} \text{ if } Z_3 = \emptyset, \\ \{q_1^{w(q_1)}, q_2^{w(q_2)}, \ldots, q_m^{w(q_m)}\} \text{ if } Z_1 - Z_3 = \emptyset, \end{cases}$$

is the axiom system of the set X.

References

[1] W. A. Pogorzelski, **Klasyczny rachunek zdań**, Warszawa 1973.

[2] H. Rasiowa, **Wstęp do matematyki współczesnej**, Warszawa 1968.

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