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Marian Maduch

ON LINDENBAUM'S ALGEBRAS OF FINITE IMPLICATIONAL ŁUKASIEWICZ'S LOGICS*

$k+1$ – valued implicational Łukasiewicz's logic (k being a nonnegative integer) is a system $S_k = \langle \underline{L}, Cn_k \rangle$ where $\underline{L} = \langle L, \rightarrow, T \rangle$ is a sentential implicational language with designated formula $T = p \rightarrow p$ and Cn_k is the consequence operation on L based on the rule of detachment $\alpha, \alpha \rightarrow \beta / \beta$ and an infinite set of the axioms (see [2]):

ax 1 $\alpha \rightarrow (\beta \rightarrow \alpha)$
ax 2 $(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))$
ax 3 $((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow ((\beta \rightarrow \alpha) \rightarrow \alpha)$
ax 4 $((\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \alpha)) \rightarrow (\beta \rightarrow \alpha)$
ax 5 $(\alpha \rightarrow (\alpha \rightarrow^k \beta)) \rightarrow (\alpha \rightarrow^k \beta)$

$\alpha, \beta, \gamma \in L$, $\alpha \rightarrow^0 \beta = \beta$ and $\alpha \rightarrow^{i+1} \beta = \alpha \rightarrow (\alpha \rightarrow^i \beta)$.

X is S_k – theory, $X \in TH$, iff $X \subseteq L$ and $Cn_k(X) = X$. X is irreducible S_k – theory, $X \in TH_*$, iff $X \in TH$ and for every nonempty family $R \subseteq TH$ not containing X we have $\bigcap R \neq X$ (see [1]). TH_* is a basis of TH i.e. $X = \bigcap\{Y : X \subseteq Y \in TH_*\}$ for any $X \in TH$. A sentential logic $\langle \underline{L}, Cn \rangle$ such that $Cn_k(X) \subseteq Cn(X)$ for any $X \subseteq L$ is said to be an implicational strengthening of S_k.

$k+1$ – valued implicational Łukasiewicz's matrix is the algebra $\underline{A}_k = \langle A_k, \rightarrow, k \rangle$, where $A_k = \{0, 1, \ldots, k\}$ and $a \rightarrow b = min(k, k-a+b)$ for $a, b \in A_k$. Let $S_k^* = \langle \underline{L}, Cn_k^* \rangle$ and $TAUT_k = Cn_k^*(\emptyset)$ where Cn_k^* is the matrix consequence operation defined an L as follows:

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*As abstract this article is not to be reviewed.

$\alpha \notin Cn_k^*(X)$ iff there exists a valuation $v : \underline{L} \mapsto \underline{A}_k$ such that $v(X) \subseteq \{k\}$ and $v(\alpha) \neq k$.

Using the suitable theses of S_k (cf. [3]) it is easy to verify that for any $X \in TH$ the relation $\underset{\sim}{X} = \{\langle \alpha, \beta \rangle : \alpha \to \beta, \beta \to \alpha \in X\}$ is a congruence of the language $\underline{L}$ and the equivalence class $|T|$ of T coincides with X. A quotient algebra $\underline{L}/\underset{\sim}{X} = \underline{L}/X = \langle L/X, \to, \bigvee_X \rangle$ where L/X is a quotient set and $\bigvee_X = |T|$ is said to be a Lindenbaum's algebra of the system S_k.

LEMMA. *For any irreducible S_k – theory $X \in TH_*$ the Lindenbaum's algebra $\underline{L}/X$ is isomorphic with some matrix $\underline{A}_l$, $l \leqslant k$.*

PROOF. Making use of suitable theses of the system S_k and irreducibility of X one can prove that the relation $\preccurlyeq$ $\preccurlyeq = \{\langle a, b \rangle : a \to b = \bigvee_X\}$ is a linear ordering on L/X, $\bigvee_X$ is the greatest element of that ordering and L/X has at most $k+1$ elements. Moreover, for any $a, b, c \in L/X$

(1) $a \preccurlyeq \bigvee_X \to a$,
(2) $a \preccurlyeq b$ implies $b \to c \preccurlyeq a \to c$,
(3) $a \to c = b \to c \neq \bigvee_X$ implies $a = b$.

Let $L/X = \{a_0, a_1, \ldots, a_l\}, l \leqslant k$, and moreover

$$a_0 \prec a_1 \prec \ldots \prec a_j \prec a_l = \bigvee_X.$$

("$a \prec b$" is short for "$a \preccurlyeq b$ and $a \neq b$"). According to the definition of $\preccurlyeq$ we have $a_i \to a_j = a_l$ for $i \leqslant j$. If $i > j$, then by (1) – (3) we obtain

$a_j \preccurlyeq a_l \to a_j \prec a_{l-1} \to a_j \prec \ldots \prec a_{i+1} \to a_j \prec a_i \to a_j$ and $a_i \to a_j \prec a_{i-1} \to a_j \prec \ldots \prec a_{j+1} \to a_j \prec a_j \to a_j = a_l$.

Hence $a_i \to a_j = a_{min(l,l-i+j)}$ for any i, j.

COROLLARY 1. (strong completeness theorem for S_k) $S_k = S_k^*$.

COROLLARY 2. (degree of maximality (see [4]) of S_k) *The only structural implicational strengthenings of S_k are logics S_l, $l \leqslant k$.*

COROLLARY 3. (degree of completeness of S_k) *The only invariant S_k – theories (i.e. S_k – theories closed with respect to substitution) are sets $TAUT_l$, $l \leqslant k$.*

Any algebra $\langle A, \to, \bigvee \rangle$ of the type $\langle 2, 0 \rangle$ fulfilling the axioms

i1. $a \to (b \to a) = \bigvee$
i2. $(a \to b) \to ((b \to c) \to (a \to c)) = \bigvee$
i3. $(a \to b) \to b = (b \to a) \to a$
i4. $(a \to b) \to (b \to a) = b \to a$
i5. $a \to (a \to^k b) = a \to^k b$
i6. $\bigvee \to a = a$

will be called an ŁI_k-algebra. Dealing with implicative filters (see [1]) instead of S_k – theories, using the lemma given above, we obtain

THEOREM. *The class of* ŁI_k *– algebras is the equational class generated by the matrix* $\underline{A}_k$: Ł$I_k = HSP(\underline{A}_k)$.

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Higher Engineering School
Opole