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UNDECIDABILITY OF SOME LOGICAL EXTENSIONS OF AJDUKIEWICZ-LAMBEK CALCULUS*

This is an abstract of a paper which will be published in Studia Logica.

Ajdukiewicz-Lambek calculus of syntactical types (AL) is a formal system in the alphabet including a finite number of individual constants, the function symbols $\cdot$, $\backslash, /$ and the relation symbol $\leqslant$. Terms and atomic formulas are defined in a natural way. There are no other formulas in the calculus.

We have the following axioms schemas:

$(a1)$ $x \leqslant x$,
$(a2)$ $x(yz) \leqslant (xy)z$'
$(a3)$ $(xy)z \leqslant x(yz)$,

where x, y, z are arbitrary terms. We use five rules of inference:

$$(r1)\ \frac{xy \leqslant z}{y \leqslant x\backslash z}, \qquad (r2)\ \frac{xy \leqslant z}{x \leqslant z/y},$$

$$(r3)\ \frac{y \leqslant x\backslash z}{xy \leqslant z}, \qquad (r4)\ \frac{x \leqslant z/y}{xy \leqslant z}, \qquad (r5)\ \frac{x \leqslant y, y \leqslant z}{x \leqslant z}.$$

AL is a decidable system [1]. However, its expressibility power is very frail. For example we cannot distinct the types denoted by two constants a, b. The natural task arises to create other systems describing the same mathematical reality but including more information. The simplest solution is to add to the calculus the apparatus of first-order logic. The question

*As abstract this article is not to be reviewed.

arises whether such logical extensions are still decidable systems. We will give a negative answer to this question for some natural class of extensions.

As a direct extension of the calculus let us consider the first-order theory L with equality and non-logical symbols same as in the calculus. L is based on the following axioms:

$(L1)$ $x \leqslant x$,
$(L2)$ $(xy)z \leqslant x(yz)$ & $x(yz) \leqslant (xy)z$,
$(L3)$ $x \leqslant y$ & $y \leqslant z \to x \leqslant z$,
$(L4)$ $xy \leqslant z \leftrightarrow x \leqslant z/y$,
$(L5)$ $xy \leqslant z \leftrightarrow y \leqslant x\backslash z$.

THEOREM 1. *The theory L is decidable.*

The proof uses essentially the weakness of AL in which we cannot express the non-triviality of the precedence relation. In first-order logic we have no such restriction. As a system more compatible with natural linguistic intuition we take the theory G_d^0 obtained from L by adding to L the axioms:

$(L6)$ $x \leqslant y$ & $y \leqslant x \to x = y$,
$(L7)$ $(Ex, y)(x \leqslant y$ & $x \neq y)$,
$(L8)$ $(Ex, y)(xy \neq yx)$,
$(G1)$ $x \leqslant y(y\backslash x)$,
$(G2)$ $x \leqslant (x/y)y$.

The theory G_d^0 is simply the theory of non-commutative non-trivially ordered groups with division operations defined naturally. The next theorem implies evidently the undecidability of G_d^0.

THEOREM 2. *The theory G^0 of non-commutative non-trivially ordered groups is undecidable.*

The system G^0 has some significance in mathematics [4]. However, from the linguistic point of view the description of the syntactical types structure as a group seems to be artificial, because it introduces new types which have no natural interpretation in a language. Thus, it is purposeful to consider other extensions of AL which do not admit to expand them to the group theory.

The basic system will be the theory L_b determined by the axioms $(L1)$ – $(L8)$, which are undoubtly adequate to the linguistic reality. Let us add to L_b two axioms:

$(L9)$ $(Ex, y)(x \nleq y(y\backslash x))$,
$(L10)$ $(Ex, y)(x \nleq (x/y)y)$,

which are simply the negations of $(G1)$ and $(G2)$ respectively. Let L_0' denotes this theory. L_b' does not allow to interpret the division operations in the group-theoretical sense. It arises from L_b by adding some existential sentences. We will call such extensions existential extensions and prove a general theorem about them.

THEOREM 3. *Every existential extension of L_b is undecidable.*

The above results do not imply the essential undecidability of L or L_b. In fact, it is not the case, because L_b and many of its extensions have finite models. It follows from the general Tarski's results [2] that L_b has an essentially undecidable finite extension. Unfortunately, we do not know any such extension which would be compatible with linguistic interpretations.

We show that there exist extensions of L_b admitting no finite model which are also decidable systems. We consider a theory L_b'' obtained by adding to L_b' new axioms:

$(L11)$ $(x)(Ey)(x < y)$,
$(L12)$ $(x)(Ey)(y < x)$.

THEOREM 4. *The theory L_b'' is undecidable but not essentially undecidable.*

The obtained results prove that the reality described by AL is too complicated to be expressed fully by means of a decidable first-order theory. We must choose between the stronger but undecidable theory and the very weak system, which, however, admits an effective method of verifying whether a sentence is valid or not. For practical reasons the linguists choose the second possibility. We hope that their choice will be more second possibility. We hope that their choice will be more explained by undecidability theorems given in this paper.

References

[1] J. Lambek, *The mathematics of sentence structure*, **American Mathematical Monthly** 65 (1958), pp. 154–170.

[2] A. Tarski, *A general method in proofs of undecidability*, [in:] A. Tarski, A. Mostowski, R. Robinson (eds.) **Undecidable theories**, North Holland Publ. Co., Amsterdam, 1953.

[3] A. Tarski, *Undecidability of the elementary theory of groups*, ibidem.

[4] L. Fuchs, **Partially ordered algebraic systems**, Pergamon Press, Oxford, 1963.

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