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A STRONGLY FINITE LOGIC WITH INFINITE DEGREE OF MAXIMALITY

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Let $\underline{M} = \{M_i | i \in I\}$ be a set of logical matrices for a propositional language $\underline{S}$, each M_i being of the form $(\underline{A}_i, B_i)$, where: $\underline{A}_i$ – an algebra similar to $\underline{S}$, B_i – the designated set. Define $C_{\underline{M}}$ by the condition: for every formula α of $\underline{S}$ and every set X of formulas

$$\alpha \in C_{\underline{M}}(X) \text{ iff } \forall i \in I \ \forall h : \underline{S} \to^{hom} \underline{A}_i \ (hX \subseteq B_i \Rightarrow h\alpha \in B_i).$$

For any $\underline{M}$, $C_{\underline{M}}$ is a structural consequence operation (cf. [2]). A consequence operation C in $\underline{S}$ is said to be *strongly finite* if there is some finite set $\underline{M}$ of finite matrices, with $C = C_{\underline{M}}$.

Let C^1, C^2 be two consequences in $\underline{S}$. We write $C^1 \leqslant C^2$ if for every set X of formulas, $C^1(X) \subseteq C^2(X)$. For any consequence operation C define the degree of maximality of C (see [4]) to be

$$card\{C^1 : C^1 \text{ is a structural consequence and } C \leqslant C^1\}.$$

It is well-known that the degree of completeness of any strongly finite consequence is finite (cf. [5]). A conjecture in [6] is that so is the degree of maximality. In this paper a counterexample to this conjecture will be given.

Consider the following implicational-negational Sugihara matrix (cf. [1]): $M_4 = (\underline{A}_4, B_4)$, where $\underline{A}_4 = (A_4, \to, \sim)$, $A_4 = \{+2, +1, -1, -2\}$, $B_4 = \{+2, +1\}$, and

$$x = -x$$

$$x \to y = \begin{cases} max(\sim x, y) & \text{if } z \leqslant y, \\ min(\sim x, y) & \text{otherwise.} \end{cases}$$

For any $a \in A_4$ define $|a|$ to be a if $a > 0$ and $-a$ otherwise. Let $\vartheta(\alpha)$ denote the set of all sentential variables occurring in α. A valuation h of formulas in M_4 is called *boolean* for if $|hp_i| = |hp_j|$ for all $p_i, p_j \in \vartheta(\alpha)$, and *non-boolean* otherwise.

Fix any infinite sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ of implicational-negational formulas with $\alpha_1 = \alpha_1(p_1, p_2)$, $\alpha_2 = \alpha_2(p_1, p_2, p_3)$, $\alpha_3 = \alpha_3(p_1, p_2, p_3, p_4)$, …, which satisfies the following conditions:

(a) α_i is a two-valued countertautology, all $i \in \omega$

(b) for any non-boolean valuation h of α_i, $h\alpha_i \in B_4$.

It results from a definability criterion given in [3] that there exist formulas satisfying both (a) and (b).

Consider the sequence $\varrho_1, \varrho_2, \varrho_3, \ldots$ of rules of inference given by the schemes:

$$\varrho_1 : \frac{\alpha_1}{p_0};\ \varrho_2 : \frac{\alpha_2}{p_0};\ \ldots\ ;\ \varrho_i : \frac{\alpha_i}{p_0};\ \ldots$$

Put C_4 to be $C_{\{M_4\}}$ and let us mark $C_{4+\varrho_i}$ to be the consequence operation arising from C_4 by strengthening C_4 with ϱ_i as a new rule of inference. $C_{4+\varrho_i}$ is structural, all $i \in \omega$, and it can be easily proved that

(A) $C_4 \leqslant C_{4+\varrho_1} \leqslant C_{4+\varrho_2} \leqslant \ldots$

On the other hand one can prove that

(B) for all substitutions ε, $\varepsilon\alpha_{i-1} \notin C_4(\alpha_i)$, all $i \geqslant 2$, and as a consequence we get $C_{4+\varrho_{i-1}} \neq C_{4+\varrho_i}$, all $i \geqslant 2$. Thus we can have (A) strengthened to the following:

(C) $C_4 < C_{4+\varrho_1} < C_{4+\varrho_2} < \ldots$

Now the following corollary immediately follows from (C):

Corollary. *The degree of maximality of the strongly finite consequence operation C_4 is infinite.*

References

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