**B**ulletin of the **S**ection of **L**ogic
Volume 5/2 (1976), pp. 54–56
reedition 2011 [original edition, pp. 54–56]

Marek Tokarz

# DEFINITIONS BY CONTEXT IN PROPOSITIONAL LOGICS

This is a summary of a lecture read at the Seminar of the Logic Department, University of Wrocław, February 1976.

It is well-known that each connective of two-valued classical propositional calculus is syntactically definable by $\rightarrow, \sim$, but neither $\vee$ nor $\wedge$ are definable in intuitionistic logic by the other connectives. In this paper we are going to characterize a new notion of syntactical definability and then give its applications to the intuitionistic logic so that both $\vee$ and $\wedge$ become definable in the new sense.

Let $\underline{S} = (S, F_1, \ldots, F_n)$ be a *propositional language*, i.e. an absolutely free algebra with $V = \{p_1, p_2, p_3, \ldots\}$ as the free generating set. Endomorphisms of $\underline{S}$ will be called *substitutions*, and they will be denoted by $\varepsilon$. Any element of $P(S) \times S$ (where $P(S)$ is the power set of $S$) is said to be a *rule* in $\underline{S}$. We shall say that a set $X \subseteq S$ is *closed* under a rule $R = (Y, A)$ if $\varepsilon Y \subseteq X$ implies $\varepsilon A \subseteq X$, for every substitution $\varepsilon$. So, in fact, we identify between *rules* and schemes of rules.

A pair $(L, \underline{R})$ is said to be a *logic* in $\underline{S}$ provided (a) $L \subseteq S$, (b) $\underline{R}$ is a set of rules, (c) $\varepsilon L \subseteq L$, all $\varepsilon$, (d) $L$ is closed under every rule in $\underline{R}$.

From now on, $\underline{S} = (S, G, F_1, \ldots, F_n)$ stands for an arbitrary, but fixed, sentential language, $G$ being supposed to be $m$-ary connective, $m \geqslant 1$.

By the *G-restriction* of $\underline{S}$ (in symbols $\underline{S}_{-G}$) we shall mean that absolutely free algebra with $V$ as the free generating set and with the operations $F_1, \ldots, F_n : \underline{S}_{-G} = (S_{-G}, F_1, \ldots, F_n)$. By the *G-restriction* of a set of rules $\underline{R}$ (in symbols $\underline{R}_{-G}$) we shall mean the set of all those rules $(X, A) \in \underline{R}$ for which $X \cup \{A\} \subseteq S_{-G}$. We mark $X_{-G} = X \cap S_{-G}$, for $X \subseteq S$.

Let $C = C(p_1, p_{m+1}, \ldots, p_s)$ and $D = D(p_1, \ldots, p_s)$ be any formulas in $S_{-G}$. Then the inscription

$$(*) \qquad C(Gp_1 \ldots p_m, p_{m+1}, \ldots, p_s) =_{df} D(p_1, \ldots, p_s),$$

is said to be a *definition by context* of the connective $G$ in the language $\underline{S}_{-G}$. We can obtain the usual definition from $(*)$ by putting $C = p_1$ and $s = m$.

If some context definition $(*)$ and some logic $(L, \underline{R})$ are fixed, then $L_{-G+G}$ is to be understood as the least subset of $S$ satisfying the following recursive conditions:

(A) $L_{-G} \subseteq L_{-G+G}$
(B) if $A \in L_{-G+G}$, then $\varepsilon A \in L_{-G+G}$, all $\varepsilon : \underline{S} \to^{hom} \underline{S}$
(C) $L_{-G+G}$ is closed under every rule in $\underline{R}_{-G}$
(D) if $A \in L_{-G+G}$ and $A^0$ arises from $A$ by replacing some subformula of the form $\varepsilon C(Gp_1 \ldots p_m, p_{m+1}, \ldots, p_s)$ by $D$ or vice versa, then $A^0 \in L_{-G+G}$.

Definition. Let $(L, \underline{R})$ be a logic in $\underline{S}$. We shall say that $G$ is *definable by context* by $F_1, \ldots, F_n$ in the logic $(L, \underline{R})$ if there exists some context definition $(*)$ for $G$ such that $L_{-G+G} = L$.

A comment is in order: The notion of the context definability is given a very concrete meaning, but it essentially depends not only on $L$ and $(*)$ but also on $\underline{R}$. So we could generally say that our definability depends on the way the logic is formalized in.

Examples. The intuitionistic sentential logic is the pair $(INT\{$modus ponens$\})$, where $INT$ is the well-known set of intuitionistic tautologies.

(1) The conjunction connective $\wedge$ is definable by context by the other intuitionistic connectives. The suitable context definition is

$$p \wedge q \to r =_{df} p \to (q \to r).$$

(2) The disjunction connective $\vee$ is also definable by context in intuitionistic logic by

$$p \vee q \to r =_{df} (p \to r) \wedge (q \to r).$$

Finally note that the above definition of definability makes it possible to generalize the notions of interpretability, common interpretability, reconstructability (strong interpretability) and common reconstructability of logical propositional systems. But that is another story.

*The Section of Logic*
*Institute of Philosophy and Sociology*
*Polish Academy of Sciences*