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STUDIES ON CHURCH'S CALCULUS*

An extended version of this abstract will appear in Reports on Mathematical Logic.

1. In Church's calculus we establish classes of equivalent formulas built from only one propositional variable p in order to obtain the theorem on existence exactly one Lindenbaum's extension for Church's system.

Moreover, we construct a class of finitely axiomatizable systems between Church's and Grzegorczyk's systems and we consider the problem of structural completeness of Church's calculus reduced to formulas formed from the variable p .

2. We use the following notations: S^C is the smallest set of well-formed formulas built by means of all propositional variables $At = \{p, q, r, p_1, p_2, \dots\}$ and the connective of implication $\rightarrow$ ($\overline{\overline{S^C}} = \aleph^0$), $At(\Phi)$ denotes the set of all propositional variables occurring in Φ , Z_2 is the set of all two-valued tautologies from S^C . The notions of consequence operations $Sb(X)$ and $Cn(R_{0*}, X)$, where $R_{0*} = \{r_0, r_*\}$ (r_0 – the modus ponens rule, r_* – the substitution rule), are known (cf. [7], [6], pp. 105–106). $\mathcal{M}$ denotes the logical matrix, $E(\mathcal{M})$ is the set of all valid formulas in this matrix (from the language S^C).

The couple $\langle R_{0*}, A_1 \rangle$ with axioms $A_1 = \{p \rightarrow p, [p \rightarrow (p \rightarrow q)] \rightarrow (p \rightarrow q), [p \rightarrow (q \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)], (p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]\}$ and primitive rules $R_{0*} = \{r_0, r_*\}$ is called Church's system (cf. [2]).

Two formulas $\Phi, \Psi \in S^C$ are equivalent in Church's system, $\Phi \equiv \Psi$, iff $\Phi \rightarrow \Psi, \Psi \rightarrow \Phi \in Cn(R_{0*}, A_1)$.

*An abstract this article is not to be reviewed.

$\langle R_0^*, A_2 \rangle$ denotes Grzegorzczuk's system with axioms $A_2 = A_1 \cup \{(p \rightarrow q) \rightarrow [p \rightarrow (p \rightarrow q)]\}$ (cf. [3], p. 102).

The set $L(Cn(R_0^*, X))$ of all Lindenbaum's extensions of the set $Cn(R_0^*, X)$ will be defined as follows: $L(Cn(R_0^*, X)) = \{Y \subseteq S^C : Cn(R_0^*, X) \subseteq Y = Cn(R_0^*, Y) \neq S^C \wedge \forall \Phi \in S^C \setminus Y Cn(R_0^*, Y) \cup \{\Phi\} = S^C\}$.

$SCpl$ is the set of all structural complete systems (cf. [5]).

3.

LEMMA 3.1.

$$\begin{aligned} \forall \{At(\Phi) = \{p\} \Rightarrow \\ ([\Phi \in Z_2 \Rightarrow \Phi \equiv p \rightarrow p \vee \Phi \equiv p \rightarrow (p \rightarrow p)] \\ \vee \Phi \equiv [p \rightarrow (p \rightarrow p)] \rightarrow (p \rightarrow p)] \\ \wedge [\Phi \notin Z_2 \Rightarrow \Phi \equiv p \vee \Phi \equiv [p \rightarrow (p \rightarrow p)] \rightarrow p \\ \vee \Phi \equiv \{[p \rightarrow (p \rightarrow p)] \rightarrow (p \rightarrow p)\} \rightarrow p]\} \end{aligned}$$

THEOREM 3.2. $\forall \Phi \in S^C \{At(\Phi) = \{p\} \Rightarrow [\Phi \in E(\mathcal{M}) \Leftrightarrow \Phi \in Cn(R_0^*, A_1)]\}$, where $\mathcal{M} = \langle \{0, 1, 2, 3\}, \{1, 2, 3\}, f^{\rightarrow} \rangle$ and

$f^{\rightarrow}$	0	1	2	3
0	1	1	1	1
1	0	1	0	0
2	0	1	2	3
3	0	1	0	2

Lemma 3.1 and Theorem 3.2 allow us to formulate and algorithm concerned equivalents formulas.

COROLLARY 3.3.

$$\begin{aligned} \forall e: At \rightarrow \{0, 1, 2, 3\} \forall \Phi \in S^C \{At(\Phi) = \{p\} \wedge e(p) = 3 \Rightarrow \\ \{\Phi \in Z_2 \Rightarrow ([h^e(\Phi) = 0 \Rightarrow \Phi \equiv p \rightarrow (p \rightarrow p)] \\ \wedge [h^e(\Phi) = 1 \Rightarrow \Phi \equiv [p \rightarrow (p \rightarrow p)] \rightarrow (p \rightarrow p)] \\ \wedge [h^e(\Phi) = 2 \Rightarrow \Phi \equiv p \rightarrow p])\} \\ \wedge \{\Phi \notin Z_2 \Rightarrow ([h^e(\Phi) = 0 \Rightarrow \Phi \equiv \{p \rightarrow (p \rightarrow p)\} \rightarrow (p \rightarrow p)] \rightarrow p] \\ \wedge [h^e(\Phi) = 1 \Rightarrow \Phi \equiv [p \rightarrow (p \rightarrow p)] \rightarrow p \\ \wedge [h^e(\Phi) = 3 \Rightarrow \Phi \equiv p])\})\} \end{aligned}$$

We say that propositional calculus $\langle R_0^*, X \rangle$ has $\mathcal{T}$ property iff the set of all two-valued implicational tautologies is the only one Lindenbaum's extension for $\langle R_0^*, X \rangle$.

THEOREM 3.4. $\langle R_{0^*}, A_1 \rangle \in \mathcal{T}$.

4. There exist denumerable many finitely axiomatizable systems between Church's and Grzegorczyk's calculus.

Let $\underline{R} = \{\langle R_{0^*}, A_1 \cup \{\alpha_n\} \rangle\}_{n \in N}$, where $\alpha_n = \underbrace{p \rightarrow (p \rightarrow \dots (p \rightarrow q) \dots)}_{n-1 \text{ times}}$, and $\mathcal{M}_n = \langle \{0, 1, 2, \dots, n+1\}, \{1, 2, \dots, n+1\}, f_n^{\rightarrow} \rangle$ $n = 1, 2, \dots$ where

$f_n^{\rightarrow}$	0	1	2	3	...	$n-1$	n	$n+1$
0	1	1	1	1	...	1	1	1
1	0	1	0	0	...	0	0	0
2	0	1	2	3	...	$n-1$	n	$n+1$
3	0	1	0	2	...	$n-2$	$n-1$	n
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
$n-1$	0	1	0	0	...	2	3	4
n	0	1	0	0	...	0	2	3
$n+1$	0	1	0	0	...	0	0	2

Let us notice that $\alpha_n \in E(\mathcal{M}_n)$ since for $v(p) = 3$ and $v(q) = n+1$ $h^v(\alpha_n) = 0$, and $\alpha_{n+1} \in E(\mathcal{M}_n)$.

THEOREM 4.1.

- a) $Cn(R_{0^*}, A_1) \subseteq \bigcap_{n \in N} E(\mathcal{M}_n)$
- b) $E(\mathcal{M}_m) \subsetneq E(\mathcal{M}_n)$ for $m > n$, where $m, n \in N$
- c) $Cn(R_{0^*}, A_1 \cup \{\alpha_{n+1}\}) \subseteq E(\mathcal{M}_n) \wedge Cn(R_{0^*}, A_1 \cup \{\alpha_{n+1}\}) \subsetneq E(\mathcal{M}_{n+1})$ for every $n \in N$.

5. S_p^C is the smallest set satisfying the conditions:

- a. $p \in S_p^C$
- b. $\varphi, \psi \in S_p^C \Rightarrow \varphi \rightarrow \psi \in S_p^C$.

THEOREM 5.1.

- a) $\langle R_{0*}, A_p \rangle \in SCpl$
- b) $\langle R, Sb(A_p) \rangle \notin SCpl$, where $A_p = Cn(R_{0*}, A_1) \cap S_p^C$.

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