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SEMANTICS OF KRIPKE'S STYLE FOR SOME MODAL SYSTEMS*

This is an abstract of the paper presented at the seminar of the Section of Logic, Institute of Mathematics Nicholas Copernicus University, held by Professor Jerzy Kotas, Toruń, January 1976. The complete version of the paper will be submitted to Studia Logica.

We use the well-known and set theoretical notation. The logical connectives will be represented by $\sim, \rightarrow, L, M$ denote negation, material implication, necessity and possibility, respectively. Formulas are represented by the capitals $A, B, C, \ldots$; the symbols $k, n, i, \ldots$ denote the natural numbers $1, 2, 3, \ldots$.

Here is list of formulas important for our purpose.

(1) $LA \rightarrow A$
(2^n) $L^nA \rightarrow L^{n+1}A$
(3^n) $M^nL^nA \rightarrow L^nA$
(4^n) $L^nA \rightarrow M^nL^{n+1}A$
(5^n) $L^nA \rightarrow M^nA$
(6^n) $L^nA \rightarrow M^nL^nA$

Let Cn denote a consequences operation determined by the modal system K of Kripke, the detachment and Gödel's rule. The present paper gives of Kripke's style for the following modal systems:

$S5_n = Cn((1), (2^n), (3^n))$
$S4_n = Cn((1), (2^n))$
$T_n^k = Cn((1), (2^k), (4^n)), \ \ (k \geqslant n)$

*As abstract this article is not to be reviewed.

$T^*_n = Cn((1), (4^n))$
$C^*_n = Cn((4^n), (5^n), (6^n))$

The Kripke's style semantics for these systems will be fixed in order to use them for an explanation of some syntactic relation holding between these systems (see [1], [2], [3], [4]). Note that $S4_1$ and $S5_1$ are the well known modal systems $S4$ and $S5$ of Lewis; $S4_n$ $(n \geqslant 2)$ constituting the family of Sobociński's modal systems. T_n and T^k_n have been considered in [1], [2], [4].

By a model structure we understand any pair $\langle W, R\rangle$, where W is anonempty set and $R \subset W \times W$. Assuming $A \subset W$ and $R \subset W \times W$ we define

$R_{|A} = \{(w, w') \in W \times W : (w, w' \in A) \wedge ((w, w') \in R)\}$
$R^{-1} = \{(w, w') \in W \times W : (w', w) \in R\}$
$R^n = \{(w, w') \in W \times W : \exists_{\{w_i\}_{i \leqslant n-1}}(((w, w_1) \in R) \wedge ((w_1, w_2) \in R) \ldots ((w_{n_1}, w) \in R))\}$

The set $\{(w, w) \in W \times W : w \in W\}$ is denoted by $\triangle$. For any model structure $\langle W, R\rangle$ and any $w \in W$ by a *tree* with top w we understand the set T_w having properties:

(i) $w \in T_w$
(ii) if $(w', w'') \in R$ and $w' \in T_w$, then $w'' \in T_{w'}$.

THEOREM 1. *$S5_n$ is determined by the class of all model structures $\langle W, R\rangle$ provided R satisfies:*

(i) $\triangle \subset R$
(ii) $R^{n+1} \subset R^n$
(iii) $R^n \subset (R^n)^{-1}$

The similar theorem for $S4_n$ looks as follows.

THEOREM 2. *$S4_n$ is determined by the class of all model structures $\langle W, R\rangle$ provided R satisfies:*

(i) $\triangle \subset R$
(ii) $R^{n+1} \subset R^n$
(iii) $\forall_w \exists_{w'}(((w, w') \in R^n) \wedge (\forall_{w'' T_{w'}}(w, w'') \in R^n) \wedge (R^n_{|T_{w'}} \subset (R^n_{|T_{w'}})^{-1}))$

Note that Theorem 2 supplies some other semantics for the systems from Sobociński's family $S4_n$ and the modal system $S4$ of Lewis.

THEOREM 3. *T_n^k $(k \geqslant n)$ is determined by the class of all model structures $\langle W, R\rangle$ provided R satisfies:*

(i) $\triangle \subset R$

(ii) $R^{k+1} \subset R^k$

(iii) $\forall_w \exists_{w'}(((w,w') \in R^n) \wedge (\forall_{w'' \in T_w}(w,w'') \in R^n) \wedge (R^{n+1}_{|T_{w'}} \subset R^n_{|T_{w'}}) \wedge$
$\wedge (R^n_{T_{w'}} \subset (R^n_{T_{w'}})^{-1}))$

THEOREM 4. *T_n^* is determined by the class of all model structures $\langle W, R\rangle$ provided R satisfies:*

(i) $\triangle \subset R$

(ii) $\forall_w \exists_{w'}(((w,w') \in R^n) \wedge (\forall_{w'' \in T_{w'}}(w,w'') \in R^n) \wedge (R^{n+1}_{|T_{w'}} \subset R^n_{T_{w'}}) \wedge$
$\wedge (R^n_{|T_{w'}} (R^n_{|T_{w'}})^{-1}))$

THEOREM 5. *C_n^* is determined by the class of all model structures $\langle W, R\rangle$ provided R satisfies:*

i) $\triangle \subset R$

ii) $\forall_w \exists_{w'}(((w,w') \in R^n) \wedge (\forall_{w'' \in T_{w'}}(w,w'') \in R^n) \wedge \wedge (R^{n+1}_{|T_{w'}} \subset R^n_{|T_{w'}}) \wedge$
$(R^n_{|T_{w'}} \subset (R^n_{|T_{w'}})^{-1}))$

For any modal system S, denote by $M^n - S$ the set of all formulas which, when preceded by M (n-times), become thesis of S. Using results contained in [1], [2], [3], [4] we obtain that between the systems T_n^*, T_n^k, $S4_n$, $S5_n$ the following relation occurs:

$$M^n - T_n^* = M^n - T_n^k = M^n - S4_n = M^n - S5_n. \tag{1}$$

Notice that the given semantics of Kripke's style for T_n^*, T_n^k, $S4_n$, $S5_n$ confirm the validity of the condition (1).

Resting on the semantics of Kripke's style for Cn_n^* (Th. 5) one can prove the following

THEOREM 6. $M^n - C_n^* = M^n - S5_n$.

Note that from Theorems 3 and 5 we have $C_n^* \subsetneq T_n$. The following theorem states that C_n^* are the least modal systems in the class of all normal *modal systems containing the Kripke's modal system K and having the property:* $M^n - C_n^* = M^n - S5_n$.

THEOREM 7. *For any normal modal system* $S \supseteq X$:
If $M^n - S = M^n - S5_n$, *then* $C_n^* \subseteq S$.

References

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