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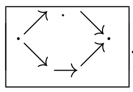
SOME PROPERTIES OF THE HIERARCHY OF MODAL LOGICS
(Preliminary report)

We are concerned with modal logics in the class EM^0 of extensions of M^0 (Von Wright's system). $\mathbf{G}$ denotes reflexive *frames.* $M\mathbf{G}$ the modal logic on $\mathbf{G}$ in the sense of Kripke. M is *finite* if $M = M\mathbf{G}$ for some finite $\mathbf{G}$. Finite $\mathbf{G}$'s will be drawn as framed diagrams, e.g. $\mathbf{G} = \boxed{\cdot \to \cdot}$; $\mathbf{G} = \boxed{\cdot \rightleftarrows \cdot}$; the latter shorter denoted by $\boxed{\cdot - \cdot}$. EM^0 is a complete lattice with *zero* M^0 and *one* $M\boxed{\cdot}$. If $M \subset M'$ (proper!) M' is a *succ*(essor) of M. An *ip* of M is an immediate predecessor of M. E.g. $M\boxed{\cdot \to \cdot}$ and $M\boxed{\cdot - \cdot}$ are the only ip's of $M\boxed{\cdot}$ in $ES4$. $M[P]$ denotes the extension of M adding P as a "new axiom", e.g. $S4 = M^0[\Box p \to \Box\Box p]$. M is finitely axiomatizable (f.a.) if $M = M^0[P]$ for some formula P. One easily shows if M is f.a. then each predecessor is separated from M by an ip of M. It is known that each finite M is f.a. and has only finitely many succ's all of which are finite.

Proposition 1. *If $M \supseteq S4$ is finite then M has only finitely many ip's all of which are finite. The same holds for the extensions of $T[\Box p \to \Box\Box p]$.*

Let $M\mathbf{C}_n$ denote the n-circular modal logic, e.g. $\mathbf{C}_5 =$ .

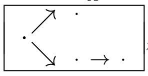
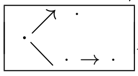
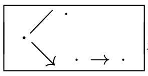
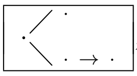
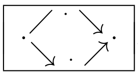
Proposition 2. *If n is prime then $M\mathbf{C}_n$ is an ip of $M\boxed{\cdot}$.*

Hence $M\boxed{\cdot}$ has infinitely many ip's in EM^0. Next we are concerned with criteria of Jankov type.

PROPOSITION 3. *$S4[P] = S5$ iff $\mathcal{P} \in S5$ and $P \notin M$* $\boxed{\cdot - \cdot}$.

Define *rank* of P by recursion: $rk\ p = 0$ (p variable); $rk\ \neg P = rk\ P$; $rk\ P \wedge Q = max\{rk\ P, rk\ Q\}$; $rk\ \Box P = rk\ P{+}1$. E.g. $rk\ (\Box p \to \Box\Diamond p) = 2$.

PROPOSITION 4. *Never $M^0[P] = S4$, $S5$ or B, if $rk\ P \leqslant 1$ (B Brouwer's system).*

PROPOSITION 5. *If $rk\ P = 2$ then $M[P] = S4$ iff $P \in S4$ and $\notin$ $M\mathbf{G}$ for each $\mathbf{G}$ =* $\boxed{\cdot \to \cdot \to \cdot}$, $\boxed{\cdot - \cdot \to \cdot}$, , , , , .

PROPOSITION 6. *If $rk\ P = 2$ then $M^0[P] = B$ iff $P \in B$ and $P \notin$ M* $\boxed{\cdot \to \cdot}$.

E.g. $B = M^0[Q]$, $Q = p \to \Box\Diamond p$ (Brouwer's axiom). By Prop. 6 for instance also $B = M^0[p \wedge \Diamond q \to \Diamond(q \wedge \Diamond p)$. The "seven graph criterion" (Prop. 5) may also be extended to a criterion for $rk\ P = 3, 4, \dots$

It is known that $S5$ has one ip in $ES4$ only. The situation completely changes if we pass to EM^0.

PROPOSITION 7. *$S4$, B, $S5$ have infinitely many ip's is EM^0.*

Let us finally state the following

CONJECTURE. All ip's of any finite M (in EM^0) are finite.

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