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ON FINITELY BASED CONSEQUENCE OPERATIONS

This is an abstract of the paper submitted to Studia Logica. The result was presented at the XII Conference on History of Logic, Cracow, July 5-9, 1976.

S. L. Bloom [2] posed the following two questions (the first of them can also be found in R. Wójcicki [8]):

(1) whether the consequence operation determined by a finite matrix is always finitely based,

(2) whether the meet of two finitely based consequence operations is always finitely based.

In this paper both of these questions are answered negatively.

§0. Let $\mathbb{T} = \langle T, d_0, \ldots, d_n\rangle$ be a free algebra of terms of some finite type free generated by an infinite set of variables $V = \{x_0, x_1, \ldots\}$. We shall use Greek lower case $\alpha, \beta, \ldots$ for terms and Greek upper case $\Phi, \Psi, \ldots$ for sets of terms. The symbol $V(\Phi)$ denotes the set of variables occurring in terms of Φ. If X is a set then the symbols $P(X), P_f(X)$ denote the set of subsets of X and the set of finite subsets of X respectively.

A mapping $C : P(T) \to P(T)$ is a consequence operation in T iff for every $\Phi, \Psi \subseteq T$, $\Phi \subseteq CC(\Phi) \subseteq C(\Phi \cup \Psi)$. A consequence operation C is algebraic (or finite) iff for every $\Phi \subseteq T$, $C(\Phi) = \bigcup CP_f(\Phi)$, a consequence operation C is structural iff for every $\Phi \subseteq T$, $\varepsilon \in Hom(\mathbb{T}, \mathbb{T})$, $\varepsilon C(\Phi) \subseteq C(\varepsilon\Phi)$. If C is structural and algebraic then it is called standard.

A rule is a subset $\varrho \subseteq P_f(T) \times T$, a rule ϱ is structural iff it is closed under substitutions i.e. for every $\varepsilon \in Hom(\mathbb{T}, \mathbb{T})$, if $\langle\Phi, \alpha\rangle \in \varrho$ then $\langle\varepsilon\Phi, \varepsilon(\alpha)\rangle \in \varrho$. A sequent is a pair $\langle\Phi, \alpha\rangle \in P_f(T) \times T$. Every sequent $\langle\Phi, \alpha\rangle$ determines a structural rule $\varrho(\Phi, \alpha) = \{\langle\varepsilon\Phi, \varepsilon(\alpha)\rangle : \varepsilon \in Hom(\mathbb{T}, \mathbb{T})\}$.

A rule ϱ is called sequential if $\varrho = \varrho(\Phi, \alpha)$ for some sequent $\langle \Phi, \alpha \rangle$. Every set of rules R determines a consequence operation C_R such that for every $\Phi \subseteq T$, $C_R(\Phi)$ is the smallest subset of T containing Φ and closed under rules of R. A consequence operation C is finitely based iff $C = C_R$ for some finite set R of sequential rules. It should be noted that the concept of finitely based consequence operation is to be credited to R. Suszko [5] who introduced it under the name of finitely formalizable consequence operation. In the sequel we shall use the following well-known results:

Theorem 0.1. (Łoś, Suszko [4]) *A consequence operation C is standard iff $C = C_R$ for some countable set R of sequential rules.*

Theorem 0.2. (Bloom [1]) *A standard consequence operation C is not finitely based iff $C = sup\{C_n\}_{n<\omega}$ for some strictly increasing sequence $C_0 < C_1 < \ldots$ of standard consequence operations.*

A matrix is a pair $\mathcal{M} = \langle \mathbb{M}, D \rangle$ where $\mathbb{M}$ is an algebra similar to $\mathbb{T}$ and D is a subset of the carrier of $\mathbb{M}$. Every matrix $\mathcal{M}$ determines a consequence operation $C_{\mathcal{M}}$ such that $\alpha \in C_{\mathcal{M}}(\Phi)$ iff for every valuation $\vartheta \in Hom(\mathbb{T}, \mathbb{M})$, if $\vartheta\Phi \subseteq D$ then $\vartheta(\alpha) \in D$. Let us note the following:

Theorem 0.3. (Łoś, Suszko [4]) *If $\mathcal{M}$ is a finite matrix then the consequence operation $C_{\mathcal{M}}$ is standard.*

Following J. Zygmunt [9] we say that a matrix $\mathcal{M}$ is proper iff for every $\vartheta \in Hom(\mathbb{T}, \mathbb{M})$, $\vartheta T \not\subseteq D$. Observe that if $\mathcal{M}$ is proper then $C_{\mathcal{M}}(\Phi) \neq T$ iff $\vartheta\Phi \subseteq D$ for some $\vartheta \in Hom(\mathbb{T}, \mathbb{M})$. This observation yields the following fact being an immediate consequence of a theorem of [9].

Theorem 0.4. (comp. Zygmunt [9]) *Let $\mathcal{M}_1, \mathcal{M}_2$ be proper matrices, then for every $\Phi \subseteq T$ the following condition holds:*

$$C_{\mathcal{M}_1 \times \mathcal{M}_2}(\Phi) = \begin{cases} T \text{ if } C_{\mathcal{M}_1}(\Phi) = T \text{ or } C_{\mathcal{M}_2}(\Phi) = T, \\ C_{\mathcal{M}_1}(\Phi) \cap C_{\mathcal{M}_2}(\Phi) \text{ otherwise.} \end{cases}$$

§1. In this section we shall give an example of finite matrix whose consequence operation is not finitely based. From now on $\mathbb{T} = \langle T, \cdot \rangle$ is assumed to be the free algebra of terms of the type $\langle 2 \rangle$ free generated by an infinite set of variables $V = \{x_0, x_1, \ldots\}$. In order to simplify the notations we adopt the convention of associating to the left and ignoring the symbol of binary operation, thus for example we shall write $\alpha\beta\gamma\alpha(\alpha\alpha)$ instead of $(((\alpha \cdot \beta) \cdot \gamma) \cdot \alpha) \cdot (\alpha \cdot \alpha)$.

Let $\mathbb{A} = \langle\{0, 1, 2\}, \cdot\rangle$ be an algebra of the type $\langle 2\rangle$ whose binary operation $\cdot$ is such that $0 \cdot 0 = 2 \cdot 2 = 2$, $1 \cdot 1 = 1$ and $a \cdot b = 0$ otherwise. Note that the set $\{0, 2\}$ is a subuniverse of $\mathbb{A}$ and the subalgebra $\mathbb{B} = \langle\{0, 2\}, \cdot\rangle$ of $\mathbb{A}$ is the two-element equivalential algebra (see [3]). Define a matrix $\mathcal{A} = \langle\mathbb{A}, \{0\}\rangle$ and a matrix $\mathcal{Z} = \langle\mathbb{B}, \{0\}\rangle$. It is easy to see that the matrices $\mathcal{A}, \mathcal{Z}$ are proper and $\mathcal{Z}$ is a submatrix of $\mathcal{A}$. We shall prove the following:

PROPOSITION 1.1. *The consequence operation $C_{\mathcal{A}\times\mathcal{Z}}$ is not finitely based.*

In preparation for the proof we need to state several lemmas. First let us observe that the consequence operation $C_{\mathcal{A}}$ is standard by virtue of Theorem 0.3 and therefore applying Theorem 0.1 we can choose a countable set R of sequential rules such that $C_{\mathcal{A}} = C_R$. For every $n = 1, 2, \ldots$ we define a set of sequential rules $R_n = \{\rho(\Phi, \alpha) : \Phi \in P_f(T),\ |V(\Phi)| \leqslant n,\ \alpha \in T = C_{\mathcal{Z}}(\Phi)\}$ and the corresponding consequence operation $C_n = C_{R\cup R_n}$. By Theorem 0.1 if follows that each the consequence operation C_n is standard and moreover we have the following:

LEMMA 1.1. $C_{\mathcal{A}} \leqslant C_n < C_{n+1}$.

PROOF. Since $R \subseteq R \cup R_n \subseteq R \cup R_{n+1}$ then $C_{\mathcal{A}} \leqslant C_n \leqslant C_{n+1}$ and it remains to show that $C_n \neq C_{n+1}$. For every $n = 1, 2, \ldots$ we define a term $\gamma_n = (x_0x_0)(x_1x_1)\ldots(x_nx_n)$. It is easy to see that $C_{\mathcal{Z}}(\gamma_n) = T$ which means that $\xi(\{\gamma_n\}, x_{n+1}) \in R_{n+1}$ and consequently $C_{n+1}(\gamma_n) = T$. On the other hand $C_{\mathcal{A}}(\gamma_n) \neq T$ (consider a valuation $\vartheta_0 \in Hom(\mathbb{T}, \mathbb{A})$ such that $\vartheta_0(x_0) = 1$ and $\vartheta_0(x_i) \neq 1$ whenever $i \neq 0$) and therefore it suffices to show that $C_n(\gamma_n) \subseteq C_{\mathcal{A}}(\gamma_n)$. We shall prove that if $|V(\Phi)| \leqslant n$ and $C_{\mathcal{Z}}(\Phi)$ then for every $\varepsilon \in Hom(\mathbb{T}, \mathbb{T})$, $\varepsilon\Phi \not\subseteq C_{\mathcal{A}}(\gamma_n)$ which means that $C_{\mathcal{A}}(\gamma_n)$ is closed under rules of R_n. Suppose that $|V(\Phi)| \leqslant n$, $C_{\mathcal{Z}}(\Phi) = T$ and $\varepsilon \in Hom(\mathbb{T}, \mathbb{T})$. Since $|V(\gamma_n)| = n + 1$ then $|V(\Phi)| < |V(\gamma_n)|$ and therefore there must exist a variable $x_c \in V(\gamma_n)$ such that for every variable $x_j \in V(\Phi)$, $\{x_c\} \neq V(\varepsilon(x_j))$. This gives that for every $x_j \in V(\Phi), x_c \notin V(\varepsilon(x_j))$ or $\{x_c\} \subsetneq V(\varepsilon(x_j))$. Let $\vartheta_c \in Hom(\mathbb{T}, \mathbb{A})$ be such that $\vartheta_c(x_c) = 1$ and $\vartheta_c(x_i) \neq 1$ whenever $i \neq c$. Then for every $x_j \in V(\Phi)$ $\vartheta_c(\varepsilon(x_j)) \in \{0, 2\}$ because every term $\varepsilon(x_j)$ must contain a variable whose value under ϑ_c is distinct from 1. Pick a valuation $\overline{\vartheta}_c \in Hom(\mathbb{T}, \mathbb{B})$ such that $\overline{\vartheta}_c(x_j) = \vartheta_c(\varepsilon(x_j))$ for every $x_j \in V(\Phi)$. Then $\overline{\vartheta}_c(\alpha) = \vartheta_c(\varepsilon(\alpha))$

whenever $V(\alpha) \subseteq V(\Phi)$ which yields that $\vartheta_c\varepsilon\Phi = \overline{\vartheta}_c\Phi \nsubseteq \{0\}$ because $C_{\mathcal{Z}}(\Phi) = T$ and the matrix $\mathcal{Z}$ is proper. The observation that $\vartheta_c(\gamma_n) = 0$ completes the proof of lemma. Q.E.D.

Note that from Lemma 1.1 follows the existence of a strictly increasing sequence of standard consequence operations beginning with a consequence operation determined by a finite matrix. Such a result (disproving a conjecture of R. Wójcicki [7]) was first obtained by M. Tokarz. The present proof of this result is a modification of the basic idea of M. Tokarz [6].

Let $C_\omega = sup\{C_n\}_{n<\omega}$, then $C_\omega = C_{R\cup\bigcup\{R_n\}_{n<\omega}}$ and from Lemma 1.1 and Theorem 0.2 it follows that C_ω is not finitely based. Thus Proposition 1.1 can be obtained as a direct consequence of the following:

Lemma 1.2. $C_{\mathcal{A}\times\mathcal{Z}} = C_\omega$.

§2. In this section it will be shown that the meet of two finitely based consequence operations need not to have any finite base.

Let us define two sequential rules: $r = \{\langle\{\alpha\}, \alpha\beta\rangle : \alpha, \beta \in T\}$ and $l = \{\langle\{\alpha\}, \beta\alpha\rangle : \alpha, \beta \in T\}$ and the corresponding two consequence operations Cr and Cl in T determined by the rules r and l respectively. We shall prove the following:

Proposition 2.1. *The consequence operation* $inf\{Cr, Cl\}$ *is not finitely based.*

First, for every $n = 0, 1, \ldots$ we define two mappings $Cr^n, Cl^n : P(T) \to P(T)$ putting for every $\Phi \subseteq T$, $\alpha \in T : \alpha \in Cr^n(\Phi)$ iff for some $m \leqslant n$ there exists a sequence of terms $\triangle_0, \ldots, \triangle_m$ such that $\triangle_m = \alpha$ and for every $i \leqslant m$, if $\triangle_i \in \Phi$ then $\triangle_i = \triangle_j\beta$ for some $j < i$, $\beta \in T$; $\alpha \in Cl^n(\Phi)$ iff for some $m \leqslant n$ there exists a sequence of terms $\triangle_0, \ldots, \triangle_m$ such that $\triangle_m = \alpha$ and for every $i \leqslant m$, if $\triangle_i \notin \Phi$ then $\triangle_i = \beta\triangle_j$ for some $j < i$, $\beta \in T$.

The proof of the following simple lemma is left to the reader:

Lemma 2.1.

(i) $Cr^n(\Phi) \subseteq Cr^{n+1}(\Phi) \subseteq Cr(\Phi), Cl^n(\Phi) \subseteq Cl^{n+1}(\Phi) \subseteq Cl(\Phi)$;

(ii) $Cr(\Phi) = \bigcup\{Cr^n(\Phi)\}_{n<\omega}, Cl(\Phi) = \bigcup\{Cl^n(\Phi)\}_{n<\omega}$;

(iii) $Cr(\Phi) = \bigcup\{Cr(\alpha : \alpha \in \Phi\}, Cl(\Phi) = \bigcup\{Cl(\alpha) : \alpha \in \Phi\}$.

For every $n = 1, 2, \ldots$ we define a matrix $\mathcal{N}_n = \langle\mathbb{N}_n, , \{0\}\rangle$ where $\mathbb{N}_n =$

$\langle \{0, \ldots, n\}, \cdot \rangle$ is an algebra of the type $\langle 2 \rangle$ whose binary operation $\cdot$ is such that $(n-1) \cdot 0 = n$, $a \cdot (a+1) = a+1$ whenever $a+1 < n$ and $a \cdot b = 0$ otherwise.

LEMMA 2.2. *If* $n = 1, 2, \ldots$, $\vartheta \in Hom(\mathbb{T}, \mathbb{N}_n)$ *and* α *is a term such that* $\vartheta(\alpha) = 0$ *then* $\vartheta Cl(\alpha) \subseteq \{0, n\}$ *and* $\vartheta Cr^{n-1}(\alpha) \subseteq \{0, \ldots, n-1\}$.

Now for every $n = 0, 1, \ldots$ we define a set of sequential rules $Q_n = \{\rho(\{\alpha, \beta\}, \gamma) : \gamma \in Cr^n(\alpha) \cap Cl^n(\beta)\}$ and the corresponding consequence operation $Cq_n = C_{Q_n}$.

LEMMA 2.3. $Cq_n < Cq_{n+1}$.

Let $Cq_\omega = sup\{Cq_n\}_{n<\omega}$, then $Cq_\omega = C_{\bigcup\{Q_n\}_{n<\omega}}$ and from Lemma 2.3 and Theorem 0.2 it follows that Cq_ω is not finitely based. Thus Proposition 2.1 follows immediately from the following:

LEMMA 2.4. $Cq_\omega = inf\{Cr, Cl\}$.

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