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THE NUMBER OF QUASIVARIETIES OF DISTRIBUTIVE LATTICES WITH PSEUDOCOMPLEMENTATION

This is an abstract of the submitted to Report on Mathematical Logic.

Following Grätzer [2] by a distributive lattice with pseudocomplementation we mean an algebra $\mathcal{A} = \langle A, \wedge, \vee, \neg, 0_{\mathcal{A}}, 1_{\mathcal{A}}\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that $\langle A, \wedge, \vee, 0_{\mathcal{A}}, 1_{\mathcal{A}}\rangle$ is a bounded distributive lattice and for every $a \in A$, $\neg a$ is the pseudocomplement of a (i.e. the greatest element of the set $\{x : x \in A, x \wedge a = 0\}$). Let $\mathbb{D}$ be the class of distributive lattices with pseudocomplementation. It is known that $\mathbb{D}$ is a variety (see [5]) and a nice characterization of the lattice of subvarieties of $\mathbb{D}$ is to be found in [3] (it is a denumerable lattice dually isomorphic to the ordinal ω^+). In this paper we will prove that the family of all subsets of a denumerably infinite set ordered by the inclusion is isomorphic to a family of quasivarieties of distributive lattices with pseudocomplementation. This result is a solution of the problem 63 of Grätzer [2] because it yields that the number of quasivarieties of distributive lattices with pseudocomplementation is $2^{\aleph_0}$.

Let $\mathcal{T} = \langle T, \wedge, \vee, \neg, 0, 1\rangle$ be the free algebra of terms of type $\langle 2, 2, 1, 0, 0\rangle$ free-generated by a denumerably infinite set of variables $V = \{z_0, z_1, \ldots\}$. By an identity we mean an expression of the form $\alpha \equiv \beta$ where α and β are terms of T. The symbol Id denotes the set of all identities. By an implication we mean an expression of the form $X \to \alpha \equiv \beta$ where $\alpha \equiv \beta$ is an identity and X is a finite (possibly empty) set of identities.

Given an algebra $\mathcal{A} \in \mathbb{D}$, a valuation in $\mathcal{A}$ is an arbitrary homomorphism v of the algebra $\mathcal{T}$ into $\mathcal{A}$. An identity $\alpha \equiv \beta$ is satisfied by the valuation v iff $v(\alpha) = v(\beta)$. The symbol $Id(v)$ denotes the set of all identities that are satisfied by v and $Id(\mathcal{A}) = \bigcap(Id(v) : v$ is a valuation in $\mathcal{A})$. An implication $X \to \alpha \equiv \beta$ is satisfied by the valuation v iff $X \subseteq Id(v)$

implies that $\alpha \equiv \beta \in Id(v)$. The symbol $Im(v)$ denotes the set of all implications that are satisfied by v and $Im(\mathcal{A}) = \bigcap(Im(v) : v$ is a valuation in $\mathcal{A})$.

A class $\mathbb{K}$ of algebras of the same type is a variety iff for some set of identities X, $\mathbb{K} = \{\mathcal{A} : X \subseteq Id(\mathcal{A})\}$. The class $\mathbb{K}$ is a quasivariety iff for some set of implications Y, $\mathbb{K} = \{\mathcal{A} : Y \subseteq Im(\mathcal{A})\}$. A characterization of quasivarieties of algebras was given by Malcev [4]. It should be noted that a quasivariety is closed under the formation of subalgebras and direct products (see [4]).

A convenient method of constructing distributive lattices with pseudocomplementation satisfying a prescribed set of implications can be obtained by transferring to lattice theory the following technique of forcing which is very familiar in logic.

Let $\mathcal{A} = \langle A, \leqslant \rangle$ be a partially ordered set. A binary relation $\Vdash \subseteq A \times T$ is called a forcing on $\mathcal{A}$ iff for every $a, b \in A$, $\alpha, \beta \in T$ the following conditions hold (see [6]):

(i) For every $z \in V$, if $a \Vdash z$ and $a \leqslant b$ then $b \Vdash z$;
(ii) $a \Vdash 1$;
(iii) $a \Vdash 0$ ($\nVdash$ denotes the complement of $\Vdash$);
(iv) $a \Vdash \alpha \wedge \beta$ iff $a \Vdash \alpha$ and $a \Vdash \beta$;
(v) $a \Vdash \alpha \vee \beta$ iff $a \Vdash \alpha$ or $a \Vdash \beta$;
(vi) $a \Vdash \neg\beta$ iff for every $b \geqslant a$, $b \nVdash \beta$.

Lemma 1. (see [6]).

(i) Every relation $\Vdash_0 \subseteq A \times V$ satisfying the condition (i) of the above definition can be extended (uniquely) to a forcing relation $\Vdash$ on $\mathcal{A}$.

(ii) For every forcing $\Vdash$ on $\mathcal{A}$, $a, b \in A$ and $\alpha \in T$ if $a \Vdash \alpha$ and $a \leqslant b$ then $b \Vdash \alpha$.

We say that an identity $\alpha \equiv \beta$ is satisfied by a forcing $\Vdash$ on $\mathcal{A}$ iff for every $a \in A$, $a \Vdash \alpha$ iff $a \Vdash \beta$. The symbol $Id(\Vdash)$ denotes the set of all identities that are satisfied by $\Vdash$ and $Id(\mathcal{A}) = \bigcap(Id(\Vdash)$: $\Vdash$ is a forcing on $\mathcal{A})$. An implication $X \to \alpha \equiv \beta$ is satisfied by $\Vdash$ iff $X \subseteq Id(\Vdash)$ implies that $\alpha \equiv \beta \in Id(\Vdash)$. The symbol $Im(\Vdash)$ denotes the set of all implications that are satisfied by $\Vdash$ and $Im(\mathcal{A}) = \bigcap(Im(\Vdash)$: $\Vdash$ is a forcing on $\mathcal{A})$.

Following Birkhoff [1] we say that a partially ordered set is inductive iff every chain of its elements has an upper bound.

LEMMA 2. *If $\mathcal{A} = \langle A, \leqslant\rangle$ is inductive, $a \in A$ and $\alpha \in T$ then for every forcing relation $\Vdash$ on $\mathcal{A}$ the following conditions are equivalent:*

(i) $a \Vdash \neg\alpha$,

(ii) for every maximal element $b \in A$ such that $a \leqslant b$, $b \nVdash \alpha$.

For every partially ordered set $\mathcal{A} = \langle A, \leqslant\rangle$ let $\Gamma(\mathcal{A})$ be the distributive lattice with pseudocomplementation of all hereditary subsets of A (see [2]). Recall that $B \subseteq A$ is hereditary iff for every $b \in B$, if $a \in A$ and $b \leqslant a$ then $a \in B$. If $H(\mathcal{A})$ is the family of all hereditary subsets of A then $\Gamma(\mathcal{A}) = \langle H(\mathcal{A}), \cap, \cup, \neg, \emptyset, A\rangle$ where for every $\Phi \in H(\mathcal{A})$, $\neg\Phi = \bigcup(\Psi : \Psi \in H(\mathcal{A}), \Psi \cap \Phi = \emptyset)$.

LEMMA 3. $Im(\mathcal{A}) = Im(\Gamma(\mathcal{A}))$.

Let N be the set of all natural numbers. It will be convenient to identity a natural number n with the set of all natural numbers that are smaller than n. For every $n \subseteq N - \{0, 1\}$ we define the corresponding implication Π_n putting:

$$\Pi_n = \neg\neg\bigvee(\neg z_i : i \in n) \equiv \bigvee(\neg z_i : i \in n) \Rightarrow$$

$$\Rightarrow 1 \equiv \bigvee(\neg(z_i \wedge \bigwedge(\neg z_j : j \in n - \{i\})) : i \in n).$$

To explain what the implication Π_n says we need the following definitions. An element a of an algebra $\mathcal{A} \in \mathbb{D}$ is called skeletal (see [2]) iff for some $b \in A$, $a = \neg b$. A finite set B of elements of an algebra $\mathcal{A} \in \mathbb{D}$ is meet-independent iff for every $C \subsetneq B$, $\bigwedge C \neq \bigwedge B$. Now we can state the following:

THEOREM 1. *If $\mathcal{A} \in \mathbb{D}$ is such that the set of all non-unit skeletal elements of $\mathcal{A}$ can be extended to a proper ideal then the following conditions are equivalent:*

(i) $\Pi_n \in Im(\mathcal{A})$,

(ii) there is no meet-independent n-element set of skeletal elements of $\mathcal{A}$ whose join also is skeletal.

For every $n \in N$ let P_n be the family of all n-element subsets of N. For every $I \subset N - \{0, 1\}$ let $S_I = \bigcup(P_n : n \in I \cup \{1\}) \cup \{N\}$. Thus, for every $I \subseteq N - \{0, 1\}$ we have the corresponding partially ordered set $\mathcal{S}_I = \langle S_I, \supset\rangle$.

It is obvious that all the maximal elements of $\mathcal{S}_I$ are singletons from P_1, the smallest element of $\mathcal{S}_I$ is N and every ascending chain of elements of $\mathcal{S}_I$ is finite which immediately yields that $\mathcal{S}_I$ must be inductive.

Lemma 4. *For every $I \subseteq N - \{0,1\}$ and $n \in N - \{0,1\}$ the following conditions are equivalent:*

(i) $\Pi_n \in Im(\mathcal{S}_I)$,

(ii) $n \in I$.

For every $I \subseteq N - \{0,1\}$ we define a set of implications $\Pi(I) = \{\Pi_n : n \in I\}$ and the corresponding quasivariety $\mathbb{K}(I) = \{\mathcal{A} : \mathcal{A} \in \mathbb{D}, \Pi(I) \subseteq Im(\mathcal{A})$.
Applying Lemma 3 and Lemma 4 we get main result of this paper:

Theorem 2. *For every $I, J \subseteq N - \{0,1\}$, $\mathbb{K}(I) \subseteq \mathbb{K}(J)$ iff $I \supseteq J$.*

Proof. Immediate, by Lemma 3 and Lemma 4. Q.E.D.

Corollary. *There exist $2^{\aleph_0}$ of quasivarieties of distributive lattices with pseudocomplementation.*

References

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