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REMARKS ON HALLDEN COMPLETENESS OF MODAL AND INTERMEDIATE LOGICS

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Let $\mathbb{T} = \langle T, \wedge, \vee, \rightarrow, \neg, L\rangle$ be the free algebra of terms of the type $2, 2, 2, 1, 1$ free generated by an infinite set of variables V. The symbol $V(\alpha)$ denotes the set of all variables occurring in the term α and TH denotes the set of all terms which are theorems of the classical propositional calculus. The following definition seems to be in accordance with the intention of E. Lemmon [2]; a set of term $\mathcal{L}$ is a modal logic iff the following conditions are satisfied:

(1) $\mathcal{L}$ is a proper subset of T and $TH \subseteq \mathcal{L}$,
(2) $\mathcal{L}$ is closed under the detachment rule i.e. for every $\alpha, \beta \in T$, if $\alpha, \alpha \rightarrow \beta \in \mathcal{L}$ then $\beta \in \mathcal{L}$,
(3) $\mathcal{L}$ is closed under the substitution rule i.e. for every $\varepsilon \in Hom(\mathbb{T}, \mathbb{T})$, if $\alpha \in \mathcal{L}$ then $\varepsilon(\alpha) \in \mathcal{L}$.

We say that a modal logic $\mathcal{L}$ is closed under the extensionality rule iff for every $\alpha, \beta \in T$, if $\alpha \rightarrow \beta$, $\beta \rightarrow \alpha \in \mathcal{L}$ then $L\alpha \rightarrow L\beta$, $L\beta \rightarrow L\alpha \in \mathcal{L}$. The symbol $\mathcal{B}$ will be reserved for the smallest modal logic closed under the extensionality rule. A modal algebra is any algebra $\mathbb{A} = \langle A, \wedge, \vee, \rightarrow, \neg, L\rangle$ of the type $\langle 2, 2, 2, 1, 1\rangle$. The reduct of a modal algebra $\mathbb{A}$ is the algebra $\mathcal{R}(\mathbb{A}) = \langle A, \wedge, \vee, \rightarrow, \neg\rangle$ obtained by discarding the modal operator L of $\mathbb{A}$. A modal matrix is any pair $\langle \mathbb{A}, D\rangle$ where $\mathbb{A}$ is a modal algebra and D is a subset of the carrier of $\mathbb{A}$. The content of the matrix $\langle \mathbb{A}, D\rangle$ will be denoted by $E\langle \mathbb{A}, D\rangle$. Recall that $E\langle \mathbb{A}, D\rangle = \{\alpha \in T$: for every $\vartheta \in Hom(\mathbb{T}, \mathbb{A})$, $\vartheta(\alpha) \in D\}$. A modal matrix $\langle \mathbb{A}, D\rangle$ will be called special if $\mathcal{R}(\mathbb{A})$ is a

Boolean algebra and D is an ultrafilter of $\mathcal{R}(\mathbb{A})$. Recall the following well-known result:

THEOREM 1. (see e.g. Hansson, Gärdenfors [1]). *For every modal logic $\mathcal{L}$ the following conditions are equivalent:*

(i) $\mathcal{B} \subseteq \mathcal{L}$

(ii) $\mathcal{L} = E\langle \mathbb{A}, D\rangle$ *for some modal matrix* $\langle \mathbb{A}, D\rangle$ *such that* $\mathcal{R}(\mathbb{A})$ *is a Boolean algebra,*

(iii) $\mathcal{L} = E\langle \mathbb{A}, D\rangle$ *for some modal matrix* $\langle \mathbb{A}, D\rangle$ *such that* $\mathcal{R}(\mathbb{A})$ *is a Boolean algebra and* D *is a proper filter of* $\mathcal{R}(\mathbb{A})$.

Following J. C. C. McKinsey [4] we say that a modal logic $\mathcal{L}$ is Hallden complete (H-complete) iff for every $\alpha, \beta \in T$ such that $V(\alpha) \cap V(\beta) = \emptyset$, if $\alpha \vee \beta \in \mathcal{L}$ then $\{\alpha, \beta\} \cap \mathcal{L} \neq \emptyset$. Let us note the well-known:

THEOREM 2. (see McKinsey [4] and Lemmon [2]). *If* $\langle \mathbb{A}, D\rangle$ *is a special modal matrix then* $R\langle \mathbb{A}, D\rangle$ *is a* H*-complete modal logic.*

E. Lemmon [2] posed the question whether the converse of the theorem above is true i.e. whether for every H-complete modal logic $\mathcal{L}$ there exists a special modal matrix $\langle \mathbb{A}, D\rangle$ such that $\mathcal{L} = E\langle \mathbb{A}, D\rangle$. In view of Theorem 1 this question is interesting only for modal logics containing $\mathcal{B}$. We shall prove that for such modal the answer is affirmative.

PROPOSITION 1. *For every modal logic* $\mathcal{L}$ *such that* $\mathcal{B} \subseteq \mathcal{L}$ *the following conditions are equivalent:*

(i) $\mathcal{L}$ *is* H*-complete,*

(ii) $\mathcal{L} = E\langle \mathbb{A}, D\rangle$ *for some special modal matrix* $\langle \mathbb{A}, D\rangle$.

PROOF. We need only to show that (i) implies (ii). Since $\mathcal{B} \subseteq \mathcal{L}$ than by Theorem 1 it follows that there exists a modal matrix $\langle \mathbb{A}, D\rangle$ such that $\mathcal{R}(\mathbb{A})$ is a Boolean algebra, D is a proper filter of $\mathcal{R}(\mathbb{A})$ and $\mathcal{L} = E\langle \mathbb{A}, D\rangle$. Let $\{D_i : i \in I\}$ be the family of all ultrafilters of $\mathcal{R}(\mathbb{A})$ containing the filter D. Then $D = \bigcap\{D_i : i \in I\}$ and consequently $\mathcal{L} = \bigcap\{E\langle \mathbb{A}, D_i\rangle : i \in I\}$. For every formula $\alpha \in T$ we put $ver(\alpha) = \{i \in I : \alpha \in E\langle \mathbb{A}, D_i\rangle\}$ and $ref(\alpha) = I - ver(\alpha)$. Then assuming H-completeness of $\mathcal{L}$ it is easy to infer that the family $\{ref(\alpha) : \alpha \in T - \mathcal{L}\}$ has the finite intersection property. Now let $\triangle$ be an ultrafilter of the Boolean algebra of all subsets of I such that $\{ref(\alpha) : \alpha \in T - \mathcal{L}\} \subseteq \triangle$. Then it is easy to check that the ultraproduct $\prod_{\triangle}(\langle \mathbb{A}, D_i\rangle : i \in I)$ is a special modal matrix and $\mathcal{L} = E(\Pi_{\triangle}(\langle \mathbb{A}, D_i\rangle : i \in I))$. Q.E.D.

Applying the same method one obtains the following characterization of H-complete intermediate logics:

PROPOSITION 2. *For every intermediate logic $\mathcal{L}$ the following conditions are equivalent:*

(i) $\mathcal{L}$ is H-complete,
(ii) $\mathcal{L} = E(\mathbb{A})$ for some well-connected pseudo-Boolean algebra $\mathbb{A}$,
(iii) $\mathcal{L} = E(\mathbb{A})$ for some strongly compact pseudo-Boolean algebra $\mathbb{A}$.

Recall that a pseudo-Boolean algebra $\mathbb{A}$ is well-connected iff the family of non-trivial filters of $\mathbb{A}$ is closed under finite intersections. The notion of well-connected pseudo-Boolean algebra is due to Professor R. Suszko who conjuctured the equivalence of (i) and (ii) of Proposition 2. A pseudo-Boolean algebra is strongly compact iff it has the smallest non-trivial filter. It should be noted that a pseudo-Boolean algebra is strongly compact iff it is non-degenerate and subdirectly indecomposable (see [3]).

References

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[2] E. Lemmon, *A note on Hallden incompleteness*, **Notre Dame Journal of Formal Logic** 7 (1966), pp. 296–300.

[3] C. G. McKay, *On finite logics*, **Indagationes Mathematicae** 29 (1967), pp. 363–365.

[4] J. C. C. McKinsey, *Systems of modal logics which are not unreasonable in the sense of Hallden*, **The Journal of Symbolic Logic** 18 (1953), pp. 109–113.

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