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THE NUMBER OF ISOMORPHISM TYPES OF SUBDIRECTLY INDECOMPOSABLE PSEUDO-BOOLEAN ALGEBRAS

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For every $n = 1, 2, \ldots$ let $\mathcal{T}(n)$ be the number of isomorphism types of subdirectly indecompossable pseudo-Boolean algebras having n-element generating set. It is known that $\mathcal{T}(1) = \aleph_0$ (see [1]) and $\mathcal{T}(3) = 2^{\aleph_0}$ (see [4]). In this paper we answer the question in the only remaining case by proving that $\mathcal{T}(2) = 2^{\aleph_0}$. Clearly this implies that $\mathcal{T}(n) = 2^{\aleph_0}$ for every $n \geqslant 2$.

For the terminology and notation the reader is referred to [4]. We will make use of the sequence of algebras $\vartheta_0, \vartheta_1, \ldots$ and the corresponding sequence of formulas of two variables $\delta_0, \delta_1, \ldots$ that are described in [4] in detail. Interesting properties of algebras and formulas mentioned above were first observed by Gerčiu, Kuznecov [2]. In particular, from a result of [2] it follows that for every $i, j = 0, 1, \ldots$; $\delta_i \in E(\vartheta_j)$ iff $i \neq j$. For every $I \in \{0, 1, \ldots\}$ such that $|I| \geqslant 2$ we define intermediate logic $P(I) = \bigcap(E(\vartheta_i) : i \in I)$. Let $\Im_2$ be the free algebra of formulas of two variables, then we have the following:

Lemma 1.

(i) The unit element of the algebra $\Im_2/\equiv_{P(I)}$ *is join-reducible;*

(ii) $E(\Im_2/\equiv_{P(I)}) \subseteq E(\Im_2/\equiv_{P(J)})$ *iff* $I \supseteq J$.

Lemma 2. *Let $\mathcal{A}$ be a pseudo-Boolean algebra generated by a set G. If the unit element of the algebra $\mathcal{A}$ is join-reducible then the set G generates the algebra $\mathcal{A}\oplus$.*

Theorem. *The number of isomorphism types of subdirectly indecomposable pseudo-Boolean algebras having a two-element generating set is* $2^{\aleph_0}$.

Proof. Consider all the pseudo-Boolean algebras of the form $(\Im_2/\equiv_{P(I)})\oplus$, $I \subseteq \{0,1,\ldots\}$, $|I| \geqslant 2$. By Lemma 1 (ii) we get that $(\Im_2/\equiv_{P(I)})\oplus$ and $(\Im_2/\equiv_{P(J)})\oplus$ are non-isomorphic whenever $I \neq J$. By Lemma 1(i) and Lemma 2 it follows that every algebra $(\Im_2/\equiv_{P(I)})\oplus$ has a two-element generating set and it is subdirectly indecomposable because it has the smallest non-trivial filter. Q.E.D.

Remark. Applying Lemma 3 of [3] one gets the result a little bit stronger then the theorem above, namely: there exist $2^{\aleph_0}$ equational classes of pseudo-Boolean algebras that are generated by single subdirectly indecomposable algebras with two-element generating sets.

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