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ACKERMANN, TAKEUTI,AND SCHNITT: FOR HIGHER-ORDER RELEVANT LOGIC (Abstract)

It is noted in [1] that there is a close relation between Gentzen-style cut-rule and the admissibility of Ackermann's rule γ for relevant logics and theories. Thus far, γ has at most been proved, in [2], for first-order relevant logics. (Related methods are applied, in [1], to yield a new proof of elementary logic, the classical adaptation of the γ-techniques as refined in [3] having been carried out by Dunn.)

It is time to move up; at the higher-order level, the classical admissibility of Gentzen's cut-rule is the basic conjecture of Takeuti, whose verification in [4] and [5] is severely non-constructive. A relevant counterpart would be a proof of γ for a suitable higher-order logic. Such logics are worth development on their own; the relevant analysis of the *proposition* is the keystone of the enterprise, as is clear already in [6] and will be clearer in [7]; the natural generalization is to the analysis of the *propositional function*.

However, it is by no means clear *what* the natural generalization of relevant first-order logics is, on grounds examined in [3]. In particular, there are difficulties over identity; without safeguards, at least on the Leibniz definition of identity, one may prove even at the level $R2$ of second-order relevant implication such apparent fallacies of relevance as $x = y \rightarrow z = z$. (A natural proposal to cure these difficulties, which stems from a suggestion by Urguhart, is examined in [3].)

Difficulties over identity are in fact difficulties over comprehension principles. I.e., in formulating the system $R2$, we adopt all the analogues for predicate quantifiers of the first-order principles of RQ (including but restricted to universal instantiation to predicate *letters*, though not to compound predicate expressions), together with *some* instances of the comprehension principle

[C] $\exists F \forall x(Fx \leftrightarrow A)$, where F is not free in A.

(Analogous schemes are of course entertained for general n-ary F, not excluding the case $n = 0$.)

Amazingly, however, the admissibility of γ in $R2$ is not *affected*, assuming the usual *pure* second-order vocabulary, by any reasonable choice of the instances of [C] that are to be assumed as axioms. But, as we are already prepared to expect, the proof is considerably more complicated than in the first-order case. Essentially, the idea is as follows. As in [1], proof of γ reduces to a demonstration that, for every non-theorem A of $R2$, there is a *normal* $R2$-theory that does not contain A. Normality here is taken in quite a strong sense. A normal $R2$-theory must contain all theorems of $R2$ (whatever choice we havemade among potential axioms [C] and their n-ary analogues); moreover, it must respect *all* the connectives and quantifiers, being consistent and complete on negation, prime on disjunction, ω-complete on universal quantifiers, and $\exists$ prime on existential quantifiers. In particular, this means that, if T is to be $R2$-normal, it must contain, whenever it contains $\exists FA(F)$, where F is an n-ary predicate letter, a theorem $A(G)$, for and n-ary predicate *parameter* G, with the dual condition on the universal predicate quantifier.

So suppose that A is a non-theorem of $R2$. By Henkin methods (to which we may apply a nice refinement set out by Belnap in [7], and independently by Gabbay), we may build a *completely regular* $R2$ theory T_{-A}, which does not contain A and which satisfies all the conditions for normality except perhaps the requirement of negation-consistency. Next, we blow T_{-A} up into an equivalent theory T^*_{-A}, by adding *copies* of the predicate parameters of T_{-A}. How many copies we add of a given predicate parameter F (which, for simplicity, we take as monadic) is calculated as follows. Think of all the formulas Fa, where a is an individual parameter. We may think of F itself as a certain function, determined by T_{-A}, defined on all individual parameters and with values in the 3-valued *truth-set* $\{t, n, f\}$. Specifically, where Fa is in T_{-A} but its negation isn't, we think of F as having the value t at a; if both Fa and $-Fa$ are in, we think of F as having the value n at a; if $-Fa$ is in but Fa isn't in, we think of F as having the value f at a. This exhausts every possibility, since T_{-A} is negation-complete. We want F, of course, to be not three-valued but *two-valued*;

the value n arises only at points a at which T_{-A} is (perhaps) inconsistent, which is what prevents T_{-A} from having the normality we desire. So let us make a copy F_i of F for *each* function from the set of individual parameters into $\{t, f\}$ which *agress* with F wherever possible; i.e., where, considered functionally, $Fa \neq n$. Clearly, this may involve making a lot of copies; e.g., if F takes n as value denumerably many times, we shall have to make continuum many copies F_i. After doing all this copying, analogously for each predicate parameter, we now form T^*_{-A} by temporarily undoing its effect; i.e., by adding as an extra axiom, for given F and each new F_i, $\forall x(Fx \leftrightarrow F_i x)$, and analogously for each n-ary F.

Next, we use T^*_{-A} to determine a *metavaluation*, in something like the sense of [6]. Specifically, our valuation rules, for the metavaluation v from all sentences of the language of T_{-A} into $\{t, f\}$, will be as follows. Again, in describing the metavaluation on anatomic formulas, we confine ourselves, for simplicity, to the case where we have an anatomic formula $F_i a$. But F_i, as we constructed it, has already been associated with a certain function from individual parameters into $\{t, f\}$, which we may call f_i. Then, simply, let $F_i a$ be t on v just in case $f_i(a) = t$, and otherwise let $F_i a$ be f on v. (Without loss of generality, we may assume that *all* predicate parameters are associated with such (in general, n-place) functions f_i, completing the specification of v on atomic sentences.)

Now we define v on all formulas by the following recursive procedure. $v(-A) = -v(A)$ and $v(A \mathbin{\&} B) = v(A) \mathbin{\&} v(B)$ in the obvious truth-functional sense. Similarly, $v(\forall x Ax) = t$ iff $v(Ap) = t$ for each individual parameter p, and $v(\forall F AF) = t$ iff $v(AP) = t$ for each predicate parameter P. We may, of course, treat existential quantifiers and disjunction as defined. Finally, $v(A \to B) = t$ just in case both $A \to B$ is a theorem of T^*_{-A} and either $v(-A) = t$ or $v(B) = t$. (This latter move, referring truth on v not merely to truth-values of parts but to reference also to membership in some *background theory*, is at the heart of the metavaluation technique as developed in [3].) By reasonably straightforward, though still somewhat tedious, inductive argument, we show that the set of truths on v is both a normal $R2$-theory and a sub-theory of T^*_{-A}. The key point, as the reader may be amused to check, is that adding all those extra predicate parameters enables us to verify all instances of the comprehension scheme $[C]$ that we have selected as axioms of $R2$. Since, perhaps, we have not selected all such instances as axioms, he may also be amused to check how the non-axioms can perhaps turn out false on v. At any rate, we have got ourselves a normal $R2$-theory without our arbitrary non-theorem A, after which γ follows as an easy corollary. (Central, incidentally, to the reasoning above is a form of the *converse Lindenbaum lemma*, for we have shown-dual to the usual Lindenbaum lemma – that a certain complete though inconsistent theory has a normal subtheory: a lemma which, carefully characterized, may be shown to hold generally.)

We have taken $R2$ to be a second-order version of RQ. Similarly, we may form a pure type theory RT by adding n-ary predicate variables and parameters at arbitrary types. Again, we have considerable freedom in choosing comprehension axioms $[C]$, while still carrying out the argument for the admissibility of γ which was sketched above. (I *think* that the argument still goes through when extensionality axioms are added also, as I have informally convinced myself. But I have not carried it out, even informally, for the case in which *typed lambda-terms* are present; i.e., where the language is not merely categorial but lambda-categorial.)

A detailed version of the above considerations and arguments may be found in [3]. Even more details will be pressented in [7], or perhaps in a possible third volume of that work.

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