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A NEW PROOF OF STRUCTURAL COMPLETENESS OF ŁUKASIEWICZ'S LOGICS

The problem of structural completeness of the finite-valued Łukasiewicz's sentential calculi was investigated and solved in [4], [7], [6]. The present paper contains a new proof of all these results.

I. Let $\langle S; F_1, \ldots, F_n\rangle$ be a propositional language. A matrix $M = \langle |M|, |M|^*; f_1, \ldots, f_n\rangle$ of this language is embeddable in a matrix $N = \langle |N|, |N|^*; g_1, \ldots, g_n\rangle$ ($M \subseteq N$) iff there exists a monomorphism $h : M \to N$. The symbol $N \times M$ stands for the product of matrices and the structural consequence generated by M (cf. [1]) is denoted by $\overrightarrow{M}$. We know (cf. [9]) for every M, N:

(1.1) $N \subseteq M \Rightarrow \overrightarrow{M} \leqslant \overrightarrow{N}$.

(1.2) $\overrightarrow{M} = \overrightarrow{M/_\sim}$ for every congruence $\sim$ of matrix M.

Moreover, one can prove (cf. [11]) that

$$(1.3)\ \ \overrightarrow{M \times N}(X) = \begin{cases} \overrightarrow{M}(X) \cap \overrightarrow{N}(X) & \text{if } X \in Sat(M) \text{ and } X \in Sat(N) \\ S & \text{if } X \notin Sat(M) \text{ or } X \notin Sat(N) \end{cases}$$

We recall that $X \in Sat(M)$ iff there exists $v : At \to |M|$ such that $h^v(X) \subseteq |M|^*$. A consequence Cn (or a propositional calculus $\langle R, A\rangle$) is structurally complete iff every structural and permissible rule of this consequence (of this calculus) is at the same time a derivable rule of Cn (of $\langle R, A\rangle$). Let $L_{Cn(0)}$ denotes the Lindenbaum's matrix $\langle S, Cn(O); F_1, \ldots, F_n\rangle$ of a consequence Cn.

The following theorem characterize a structurally complete consequence in the set of all structural consequences:

(1.4) $Cn \in Struct \Rightarrow [Cn \in SCpl \Leftrightarrow Cn = \overrightarrow{L}_{Cn(0)}]$
(1.5) $Cn \in Struct \Rightarrow \{Cn \in SCpl \Leftrightarrow \forall_{Cn_1 \in Struct}[Cn_1(0) = Cn(0) \Rightarrow Cn_1 \leqslant Cn]\}$

Theorem (1.4) is proved in [5] and (1.5) can be found in [2]. Let Sb be the consequence operation based on the substitution rule only. It is easy to see that:

(1.6) $Cn \in Struct \cap SCpl \Rightarrow CnSb \in SCpl.$

From the finitionistic point of view by a rule of inference we mean an operation with finite set of premises. Then we must consider finitionistic structural completeness. For finitionistic structural completeness (to be denoted by $SCpl_F$) it can be proved (cf. [2]) that:

(1.7) $Cn \in Struct \Rightarrow \{Cn \in SCpl_F \Leftrightarrow \forall_{Cn_1 \in Struct \cap Fin}[Cn_1(O) \Rightarrow Cn_1 \leqslant Cn]\}$

II. In the sequel, the symbol S will denote the set of formulas built up by means of propositional variables and the following connectives (notation follows that of Łukasiewicz): C (implication), K (conjunction), E (equivalence), A (disjunction) and N (negation). The symbol S^p stands for the set of all positive formulas. All results from I will be applied to these language. Let $M_n = \langle\{0, \frac{1}{n-1}, \ldots, 1\}, \{1\}; c, a, k, e, n\rangle$ be the n-valued Łukasiewicz's matrix and let $M_n^p = \langle|M_n|, \{1\}; c, a, k, e\rangle$ be a positive reduct of this matrix. Moreover, we put $R_{0^*} = \{r_0, r_*\}$ and $R_0 = \{r_0\}$, where r_0 is the modus ponens rule and r_* is the substitution rule. It is known (cf. [8]) that for $\langle R_{0^*}, A_n\rangle$ (the n-valued Łukasiewicz's calculus) and for $\langle R_{0^*}, A_n^p\rangle$ (the positive n-valued Łukasiewicz's calculus) we have:

(2.1) $Cn(R_0, Sb(A_n) \cup X) = \overrightarrow{M}_n(X)$ for every $X \subseteq S$.
$Cn(R_0, Sb(A_n^p) \cup X) = \overrightarrow{M}_n^p(X)$ for every $X \subseteq S^p$.

From now onward, the symbol L_n will denote the Lindenbaum's matrix of $\langle R_{0^*}, A_n\rangle$ and L_n^p will be the Lindenbaum's matrix of $\langle R_{0^*}, A_n^p\rangle$. Let us define a relation $\approx_n$ on S.

(2.2) $\alpha \approx_n \beta \;\Leftrightarrow\; E\alpha\beta \in E(M_n).$

From this definition it directly follows that $\alpha \approx_n \beta$ iff $h^v\alpha = h^v\beta$ for every

$v : At \to |M_n|$. The relation $\approx_n$ is a congruence of the Lindenbaum's matrix L_n. By induction one can prove that for every $v : At \to |M_n|$ and for every formula $C^k pq$ ($C^{k+1}pq = CpC^k pq$)

(2.3) $h^v(C^k pq) = min\{1, k(1 - vp) + vq\}$

Hence we obtain that for every $v : At \to |M_n|$

(2.4) $h^v(C^{n-1}pq) = \begin{cases} 1 & \text{if } vp \neq 1 \\ vq & \text{if } vp = 1 \end{cases}$

$$h^v(C^{n-2}pq) = \begin{cases} vq & \text{if } vp = 1 \\ vp & \text{if } vp = \dfrac{n-2}{n-1} \text{ and } vq = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Since in every M_n we have $h^v(CCpqq) = max\{vp, vq\} = a(vp, vq)$ we infer from (2.4) that:

(2.5) $h^v(CCC^{n-2}pqpp) = \begin{cases} \dfrac{n-2}{n-1} & \text{if } \; vp = \dfrac{n-2}{n-1} \; \text{ and } \; vq = 0 \\ 1 & \text{if } \; vp \neq \dfrac{n-2}{n-1} \; \text{ or } \; vq \neq 0 \end{cases}$

$$h^v(CC^{n-1}C_{n-2}pqqCC^{n-1}pqq) = \begin{cases} 0 & \text{if } \; vp = \dfrac{n-2}{n-1} \; \text{ and } \; vq = 0 \\ 1 & \text{if } \; vp \neq \dfrac{n-2}{n-1} \; \text{ or } \; vq \neq 0 \end{cases}$$

III. We shall prove now the following fundamental theorem on structural completeness of Łukasiewicz's logics.

(3.1) $\overrightarrow{M_n \times M_2} \in SCpl$ for every $n \geqslant 2$.

PROOF. Suppose that $n \geqslant 2$ is a fixed natural number. For every $x \in |M_n|$ let α_x^0, α_x^1 be two formulas defined as follows:

$\alpha_1^1 = Cpp$, $\alpha^1_{\frac{n-2}{n-1}} = CCC^{n-2}pqpp$,

$\alpha_0^1 = CC^{n-1}C^{n-2}pqqCC^{n-1}pqq$ (if $n > 2$),

$\alpha_x^1 = C^{x(n-1)}\alpha^1_{\frac{n-2}{n-1}}\alpha_0^1$ for $0 \leqslant x < \frac{n-2}{n-1}$, $\alpha_x^0 = N\alpha^1_{1-x}$.

From (2.3) and (2.5) it follows that for every $v : At \to |M_n|$

$$h^v(\alpha_x^i) = \begin{cases} i & \text{if } \; vp \neq \dfrac{n-2}{n-1} \; \text{ or } \; vq \neq 0 \\ x & \text{if } \; vp = \dfrac{n-2}{n-1} \; \text{ and } \; vq = 0 \end{cases}$$

It is easy to verify that for every $v : At \to |M_n|$

$h^v(C\alpha^i_x\alpha^j_y) = h^v(\alpha^{c(i,j)}_{c(x,y)})$, $h^v(A\alpha^i_x\alpha^j_y) = h^v(\alpha^{a(i,j)}_{a(x,y)})$,

$h^v(K\alpha^i_x\alpha^j_y) = h^v(\alpha^{k(i,j)}_{k(x,y)})$, $h^v(E\alpha^i_x\alpha^j_y) = h^v(\alpha^{e(i,j)}_{e(x,y)})$,

$h^v(N\alpha^i_x) = h^v(\alpha^{n(i)}_{n(x)})$.

Hence, $C\alpha^i_x\alpha^j_y \approx_n \alpha^{c(i,j)}_{c(x,y)}$, $A\alpha^i_x\alpha^j_y \approx_n \alpha^{a(i,j)}_{a(x,y)}$,

$K\alpha^i_x\alpha^j_y \approx_n \alpha^{k(i,j)}_{k(x,y)}$, $E\alpha^i_x\alpha^j_y \approx_n \alpha^{e(i,j)}_{e(x,y)}$, $N\alpha^i_x \approx_n \alpha^{n(i)}_{n(x)}$.

From the above it follows that a mapping $f : |M_n| \times |M_2| \to |L_n/\approx_n|$ defined as follows $f(\langle x, i\rangle) = [\alpha^i_x]$ is a monomorphism. Hence, $M_n \times M_2 \subseteq L_n/\approx_n$ and from (2.1) and (2.2) we obtain $\overrightarrow{M_n \times M_2} \geqslant \overrightarrow{L_n/\approx_n} = \overrightarrow{L_n}$. On the other hand, since $\overrightarrow{M_n}(0) = \overrightarrow{L_n}(0) = \overrightarrow{M_2 \times M_n}(0)$ we infer from (1.4) and (1.5) that $\overrightarrow{M_n \times M_2} \leqslant \overrightarrow{L_n}$. Hence $\overrightarrow{M_n \times M_2} = \overrightarrow{L_n} \in SCpl$.

Note that this theorem is proved without McNaughton's criterion and the representation theorem for Łukasiewicz's algebras. Theorem (3.1) can be immediately deduced from results of [3] and [10]. From (1.3) it follows that

$$\overrightarrow{M_n \times M_2}(X) = \begin{cases} \overrightarrow{M_n}(X) & \text{if} \quad X \in Sat(M_2) \\ S & \text{if} \quad X \notin Sat(M_2). \end{cases}$$

It is easy top see that $\overrightarrow{M_n \times M_2} \neq \overrightarrow{M_n}$ (for example, if $\alpha = C^{n-1}CCNpppNCCNppp$, then $\overrightarrow{M_n}(\alpha) \neq S$ and $\alpha \notin Sat(M_2)$) for $n > 2$. From this and (1.7) it follows that (cf. [7]):

(3.2) $\langle R_0, Sb(A_n)\rangle \notin SCpl_F$ for every $n > 2$.

Now we shall prove the Tokarz's theorem on structural completeness of the Łukasiewicz's sentential calculi.

(3.3) $\langle R_{0^*}, A_n\rangle \in SCpl_F$ for every $n \geqslant 2$.

Proof. To prove that $\overrightarrow{M_n}(Sb(X)) = \overrightarrow{M_n \times M_2}(Sb(X))$ for every $X \subseteq S$ it suffices to show that $\overrightarrow{M_n}(Sb(X)) = S$ for every $Sb(X) \notin Sat(M_2)$. Suppose that it is not true; i.e. there exists $Sb(X) \notin Sat(M_2)$ such that $\overrightarrow{M_n}(Sb(X)) \neq S$. Hence there exists $v : At \to |M_n|$ such that $h^v(Sb(X)) \subseteq$

$\{1\}$. Let $e : At \to S$ be a substitution defined as follows: $e(\gamma) = C\gamma\gamma$. We have $w = h^v e : At \to \{0,1\}$ so that $h^w(Sb(X)) = h^v(h^e(Sb(X))) \subseteq h^v(Sb(X)) \subseteq \{1\}$. Contradiction. Hence: $\overrightarrow{M_n \times M_2}(Sb(X)) = \overrightarrow{M_n}(Sb)) = Cn(R_0, Sb(A_n) \cup Sb(X)) = Cn(R_{0^*}, A_n \cup X)$. From (1.6) and (3.1) follows that $\langle R_{0^*}, A_n\rangle \in SCpl$.

We shall complete the paper by the following theorem:

(3.4) $\langle R_0, Sb(A^p_n)\rangle \in SCpl$ for every $n \geqslant 2$.

Proof. The proof is similar to that (3.1). Using the notation from (3.1) we have $\alpha^1_x \in S^p$ for every $x \in |M_n|$ and a mapping $g(x) = [\alpha^1_x]$, $g : |M_n| \to S^p/\approx_n$ is a monomorphism of M^p_n and L^p_n. Hence, $\overrightarrow{M^p_n} \geqslant \overrightarrow{L^p_n/\approx_n} = \overrightarrow{L^p_n}$. From (1.4) and (1.5) we obtain $\overrightarrow{M^p_n} \leqslant \overrightarrow{L^p_n}$. Thus $\overrightarrow{M^p_n} = \overrightarrow{L^p_n} \in SCpl$.

This theorem is a generalization of some theorem from [6], where it is proved that pure implicational Łukasiewicz's calculi are $SCpl_F$.

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