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INTUITIONISTIC SYSTEM WITHOUT CONTRACTION

The paper deals with a first order intuitionistic predicate calculus without contraction. We examine the Hilbert variant of this calculus and an equivalent Genzen variant of it, prove that this calculus is decidable, evaluate its complexity of deduction and study its connection with the classical logics.

Let us first describe the Hilbert-type calculus JH.

$ax.1$ $\varphi \supset (\psi \supset \varphi)$
$ax.2$ $(\varphi \supset \psi) \supset ((\psi \supset \chi) \supset (\varphi \supset \chi))$
$ax.3$ $(\varphi \supset (\psi \supset \chi)) \supset (\psi \supset (\varphi \supset \chi))$
$ax.4$ $(\varphi \supset \neg\psi) \supset (\psi \supset \neg\varphi)$
$ax.5$ $\varphi \supset (\neg\varphi \supset \psi)$
$ax.6$ $\varphi_i \supset \varphi_1 \vee \varphi_2, \quad (i = 1, 2)$
$ax.7$ $(\varphi \supset \chi) \supset ((\psi \supset \chi) \supset (\varphi \vee \psi \supset \chi))$
$ax.8$ $\varphi \supset (\psi \supset \varphi \wedge \psi)$
$ax.9$ $(\varphi \supset (\psi \supset \chi)) \supset (\varphi \wedge \chi \supset \chi)$
$ax.10$ $\forall\xi\varphi \supset \varphi(\tau|\xi)$
$ax.11$ $\varphi(\tau|\xi) \supset \exists\xi\varphi$

are JH-system axioms.

Rules of inference:

$$(MP)\ \frac{\varphi,\ \varphi \supset \psi}{\tau};\quad (R_1)\ \frac{\varphi \supset \psi(x|\xi)}{\varphi \supset \forall\xi\psi};\quad (R_2)\ \frac{\psi(x|\xi) \supset \varphi}{\exists\xi\psi \supset \varphi}$$

where $\varphi,\ \psi$ and χ are formulae, τ is a term, ξ – a bound variable, x – a free variable which is not present in the conclusions of rules (R_1) and (R_2).

One may easily prove the following theorem:

Theorem 1. *The formula $\varphi \supset \varphi \wedge \varphi$ added as an axiom to the system given the intuitionistic predicate calculus.*

Denote $\varphi + \psi = \neg\varphi \supset \neg\neg\psi$, and assume that

$(i1)$ 1. $\varphi = \neg\neg\varphi$	$(j1)$ $\varphi^1 = \neg\neg\varphi$
$(i2)$ 2. $\varphi = \varphi + \varphi$	$(j2)$ $\varphi^2 = \varphi \wedge \varphi$
$(i3)$ $(n+1)\ \varphi = n \cdot \varphi + \varphi,\ (n \geqslant 2)$	$(j3)$ $\varphi^{n+1} = \varphi \wedge \varphi^n\ (n \geqslant 2)$

For every natural number n we define formulae $\varphi^{(-n)}$ and $\varphi^{(+n)}$ as follows:
Let δ denote any of the two signs "+" and "–" (see [1]).
Assume $\varphi^{(-n)} = \varphi$ and $\varphi^{(+n)} = \neg\neg\varphi$, if φ is an atomic formula.

$$(\neg\varphi)^{(\delta n)} = \begin{cases} \neg\varphi^{(+n)}; & \text{if } \delta = \text{“}-\text{”} \\ \neg\varphi^{(-n)}; & \text{if } \delta = \text{“}+\text{”} \end{cases}$$

$$(\varphi \supset \psi)^{(\delta n)} = \begin{cases} \neg\varphi^{(+n)} \vee \psi^{(-n)}, & \text{if } \delta = \text{“}-\text{”} \\ \varphi^{(-n)} \supset \neg\neg\psi^{(+n)}, & \text{if } \delta = \text{“}+\text{”} \end{cases}$$

$$(\varphi \vee \psi)^{(\delta n)} = \begin{cases} \varphi^{(-n)} \vee \psi^{(-n)}, & \text{if } \delta = \text{“}-\text{”} \\ \varphi^{(+n)} + \psi^{(+n)}, & \text{if } \delta = \text{“}+\text{”} \end{cases}$$

$$(\varphi \wedge \psi)^{(\delta n)} = \begin{cases} \varphi^{(-n)} \wedge \psi^{(-n)}, & \text{if } \delta = \text{“}-\text{”} \\ \neg(\neg\neg\neg\psi^{(+n)} \vee \neg\neg\neg\psi^{(+n)}), & \text{if } \delta = \text{“}+\text{”} \end{cases}$$

$$(\exists\xi\varphi)^{(\delta n)} = \begin{cases} \exists\xi\varphi^{(-n)}, & \text{if } \delta = \text{“}-\text{”} \\ n \cdot (\exists\xi(\neg\neg\varphi^{(+n)}), & \text{if } \delta = \text{“}+\text{”} \end{cases}$$

$$(\forall\xi\varphi)^{(\delta n)} = \begin{cases} (\forall\xi\varphi^{(-n)})^n, & \text{if } \delta = \text{“}-\text{”} \\ \forall\xi(\neg\neg\varphi^{(+n)}), & \text{if } \delta = \text{“}+\text{”} \end{cases}$$

If is a formula of the propositional language, then let $\varphi^{(+n)}$ be $\varphi^{(+)}$.

Now we shall formulate three theorems and sketch their proofs.

Theorem 2.

a) A formula φ is deducible in the classical predicate calculus iff there exists a natural number n such that $\varphi^{(+n)}$ is deducible in JH.

b) If a formula φ, is deducible in the classical propositional calculus, then $\varphi^{(+)}$ is deducible in JH (i.e. in the intuitionistic system).

Here by deduction we mean a tree-like deduction. The deduction $\Gamma \vdash \varphi$ in JH may be defined also as the deduction $\Gamma \vdash^* \varphi$ in Ł$^*H$ (see [1]).

Assume that a formula φ is deducible in JH. Denote a length of the formula φ by $l(\varphi)$. $S(\varphi)$ is a minimal number of the vertices of trees, which proves the formula φ in JH.

THEOREM 3. *There exists such a constant c that for any deducible formula φ in JH the following inequality is true:*

$$S(\varphi) \leqslant c \cdot l^2(\varphi) \cdot 2^{l(\varphi)}$$

NOTE. It is unknown whether this inequality is true in the intuitionistic propositional calculus or not.

THEOREM 4. *If the calculus JH does not contain functions, then it is decidable.*

We shall need the sequentional variant of JH (denote it GH) to prove the theorem.

The axiom of JG is $\varphi \to \varphi$, where φ is an arbitrary formula.

Logical rules are:

$$(\supset \to)\ \frac{\varphi, \Gamma_1 \to \Pi \qquad \psi, \Gamma_2 \to \Pi}{\varphi \supset \psi, \Gamma_1, \Gamma_2 \to \Pi} \qquad (\to \supset)\ \frac{\varphi, \Gamma \to \psi}{\Gamma \to \varphi \supset \psi}$$

$$(N \to)\ \frac{\varphi, \Gamma \to \Pi \qquad \psi, \Gamma \to \Pi}{\varphi \vee \psi, \Gamma \to \Pi} \qquad (\to \vee)\ \frac{\Gamma \to \varphi_i}{\Gamma \to \varphi_1 \vee \varphi_2}\ (i = 1, 2)$$

$$(\wedge \to)\ \frac{\varphi, \psi, \Gamma \to \Pi}{\varphi \wedge \psi, \Gamma \to \Pi} \qquad (\to \wedge)\ \frac{\Gamma_1 \to \varphi \qquad \Gamma_2 \to \psi}{\Gamma_1, \Gamma_2 \to \varphi \wedge \psi}$$

$$(\neg \to)\ \frac{\varphi, \Gamma \to}{\Gamma \to \neg\varphi} \qquad (\to \neg)\ \frac{\Gamma \to \varphi}{\neg\varphi, \Gamma \to}$$

$$(\forall \to)\ \frac{\varphi(\tau|\xi), \Gamma \to \Pi}{\forall\xi\varphi, \Gamma \to \Pi} \qquad (\to \forall)\ \frac{\Gamma \to \forall(x|\xi)}{\Gamma \to \forall\xi\varphi}$$

$$(\exists \to)\ \frac{\varphi(x|\xi), \Gamma \to \Pi}{\exists\xi\varphi, \Gamma \to \Pi} \qquad (\to \exists)\ \frac{\Gamma \to \varphi(\tau|\xi)}{\Gamma \to \exists\xi\varphi}$$

Structural rules are:

$$(int \to)\ \frac{\Gamma_1, \psi, \varphi, \Gamma_2 \to \Pi}{\Gamma_1, \varphi, \psi, \Gamma_2 \to \Pi} \qquad (\to int)\ \frac{\Gamma \to \Pi_1, \psi, \varphi, \Pi_2}{\Gamma \to \Pi_1, \varphi, \psi, \Pi_2}$$

$$(ad \to)\ \frac{\Gamma \to \Pi}{\varphi, \Gamma \to \Pi} \qquad (\to ad)\ \frac{\Gamma \to}{\Gamma \to \varphi}$$

$$(cut)\ \frac{\Gamma_1 \to \varphi \qquad \varphi, \Gamma_2 \to \Pi}{\Gamma_1, \Gamma_2 \to \Pi}$$

where $\varphi, \psi, \varphi_1, \varphi_2$ are formulae, τ is a term, $\Gamma, \Gamma_1, \Gamma_2, \Pi, \Pi_1, \Pi_2$ are consequents of the formulae, x is a free variable which does not occur in lower sequent of the rules $(\to \forall)$ and $(\exists \to)$, ξ is a bound variable.

Lemma 1.

a) A sequent $\Gamma \to \varphi$ is deducible in JG iff $\Gamma \vdash \varphi$ in JH.

b) A sequent $\Gamma \to$ is deducible in JG iff $\Gamma \vdash \neg(F \supset F)$ in JH.

The proof is analogous to that of Proposition 1 of [2].

Assume that $\Pi_n = \varphi_1, \varphi_2, \ldots, \varphi_n$, $(n \geqslant 0)$ is a consequent of formulas. Let $\Pi_n^{(-m)}$, $\Pi_n^{(+m)}$ and $\Pi_n^{(+)}$ (see [1]) be defined as follows:

$\Pi_n^{(-m)} = \varphi_1^{(-m)}, \varphi_2^{(-m)}, \ldots, \varphi_n^{(-m)}$
$\Pi_n^{(+m)} = \varphi_1^{(+m)}, \varphi_2^{(+m)}, \ldots, \varphi_n^{(+m)}$
(k_1) $\Pi_0^{[+]} = \neg(F \supset F)$
(k_2) $\Pi_1^{[+]} = \neg\neg\varphi_1$
(k_3) $\Pi_{n+1}^{[+]} = \neg\varphi_{n+1} \supset \Pi_n^{[+]}$, $(n \geqslant 1)$

Lemma 2. *Assume that $m \geqslant n$. Then:*

(a) if $JG \vdash \varphi \to \psi$, then $JG \vdash m \cdot \varphi \to n \cdot \psi$

(b) if $JG \vdash \varphi \to \psi$, then $JG \vdash \varphi^n \to \psi^m$

(c) $JG \vdash \varphi^{(+m)} \to \varphi^{(+n)}$ and $JG \vdash \varphi^{(-n)} \to \varphi^{(-m)}$

Lemma 3. *The sequent $\Gamma \to \Pi$ is deducible in the classical sequentional calculus if there exists a natural number n such that the sequent $\Gamma^{(-n)} \to (\Pi^{(+n)})^{[+]}$ is deducible in JG.*

Lemma 3. *The cut may be eliminated in JG*

Lemmas 1 and 2 yield Theorem 2.

Lemmas 3 and 4 yield Theorem 4.

The proof of Theorem 3 is analogous to that Theorem 1 (see [2]).

References

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