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FUZZY PROPOSITIONAL LOGIC (AN ALGEBRAIC APPROACH)

This is an abstract of the paper submitted to Studia Logica.

The present paper contains some technical results on a many-valued logic with truth values from the interval of real numbers $[0, 1]$. This logic, discussed originally in [1], latter in [2] and [3], was called the logic of fuzzy concepts. Our aim is to give an algebraic axiomatics for fuzzy propositional logic. For this purpose the variety of L-algebras (see [4]) with signature enriched with a unary operation – involution (called here LI-algebras) is studied. A one-to-one correspondence between congruences on an LI-algebra and filters of a special kind (Theorem 1) is used to prove the representation theorem for LI-algebras. By this theorem every LI-algebra is isomorphic to a subdirect product of chains. The full characteristic of the subdirectly irreducible LI-algebras is given (Theorem 4). It turns out that the variety of all L-algebras, as well as any of its subvarieties, is generated by its finite algebras.

A variety v is called *critical* if for any subvariety $v' \subseteq v$ the following two conditions are equivalent:

(a) $v' \neq v$
(b) v is generated by a suitable finite algebra from v'

We prove that among all the subvarieties of the variety of all LI-algebras, there exists only one critical (Theorem 5.)

In [2] and [3] a fuzzy propositional logic is defined as a set $E(\mathcal{L})$ of formulas valid in the model

$$\mathcal{L} = \langle [0, 1], \vee, \wedge, \mapsto, \sim \rangle$$

where the operations $\vee, \wedge, \mapsto, \sim$ are defined in the following way: for $a, b \in [0,1]$

$$\begin{array}{rcl} a \vee b & = & max(a,b) \qquad a \wedge b \; = \; min(a,b) \\ a \mapsto b & = & \left\{ \begin{array}{ll} a \leqslant b, & 1 \\ a > b, & 0 \end{array} \right. \qquad \sim a \; = \; 1 - a \end{array}$$

(1 − is the designated element).

Let us consider the model $\mathcal{L}^+ = \langle [0,1], \vee, \wedge, \rightarrow, \neg, \sim \rangle$ where the operations $\vee, \wedge, \rightarrow, \neg, \sim$ are defined as follows: for $a, b \in [0,1]$

$$\begin{array}{rcl} a \vee b & = & max(a,b) \qquad a \wedge b \; = \; min(a,b) \\ a \rightarrow b & = & \left\{ \begin{array}{ll} a \leqslant b, & 1 \\ a > b, & b \end{array} \right. \qquad \neg a \; = \; \left\{ \begin{array}{ll} a = 0, & 1 \\ a > 0, & 0 \end{array} \right. \\ a & = & 1 - a \end{array}$$

Proposition 1. $E(\mathcal{L}) = E(\mathcal{L}^+)$.

The proof follows from the fact, that $(\vee, \wedge, \mapsto, \sim)$ and $(\vee, \wedge, \rightarrow, \neg, \sim)$ are equivalent bases. Indeed, it is easy to verify that for $a, b \in [0,1]$

$$\begin{array}{rcl} a \rightarrow b & = & (a \mapsto b) \vee b \qquad \neg a \; = \; a \mapsto \sim (a \mapsto a) \\ a \mapsto b & = & \neg \sim (a \rightarrow b) \end{array}$$

Now we introduce the notion of an *LI*-algebra. An *LI*-algebra is the algebra $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, \neg, \sim \rangle$, where $\langle A, \vee, \wedge, \rightarrow, \neg \rangle$ is a Brouwerian (alias pseudo-Boolean) algebra and for $a, b \in A$

(1) $(a \rightarrow b) \vee (b \rightarrow a) = 1$
(2) $a \vee b = \sim (\sim a \wedge \sim b)$
(3) $\sim\sim a = a$
(4) $\neg \sim (a \rightarrow b) \leqslant \sim b \rightarrow \sim a$
(5) $\neg \sim a \leqslant a$
(6) $\sim \neg a = \neg\neg a$
(7) $\neg \sim (a \vee b) = \neg \sim a \vee \neg \sim b$

Since an *LI*-algebra is based on Brouwerian algebra, one can change inequalities (4) and (5) into the appropriate equalities.

A filter ∇ in a n *LI*-algebra is said to be *involutive* if $a \in \nabla \Rightarrow \neg \sim a \in \nabla$.

THEOREM 1. *Let $\mathcal{A}$ be an LI-algebra. Then there exists a one-to one correspondence between the congruences on $\mathcal{A}$ and the involutive filters in $\mathcal{A}$.*

An involutive filter ∇ in the LI-algebra $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, \neg, \sim\rangle$ is said to be *maximal* provided that

(i) ∇ is proper ($\nabla \neq A$)
(ii) for any involutive filter ∇' if $\nabla \subseteq \nabla'$, then either $\nabla' = \nabla$ or $\nabla' = A$.

Let $\ulcorner a$ be abbreviation for $\sim \neg \sim a$.

THEOREM 2. *For every involutive filter ∇ in the LI-algebra $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, \neg, \sim\rangle$ the following conditions are equivalent:*

(j) ∇ is maximal
(jj) ∇ is prime
(jjj) for any $a \in A$ either $a \in \nabla$ or $\ulcorner a \in \nabla$.

THEOREM 3. *The intersection of all maximal involutive filters in an LI-algebra $\mathcal{A}$ is the unit filter $\{1\}$.*

We shall say that the LI-algebra $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, \neg, \sim\rangle$ is a *chain* if for any $a, b \in A$ either $a \leqslant b$ or $b \leqslant a$.

REPRESENTATION THEOREM. *Every LI-algebra $\mathcal{A}$ is isomorphic to a subdirect product of the chains $\mathcal{A}/\nabla_i$, where $\{\nabla_i\}_{i \in I}$ is the family of all maximal involutive filters in $\mathcal{A}$.*

The variety of all LI-algebras will be denoted by ϑ_{LI}.

THEOREM 4. *For $\mathcal{A} \in \vartheta_{LI}$ the following conditions are equivalent:*

(a) $\mathcal{A}$ is subdirectly irreducible
(b) $\mathcal{A}$ is simple (has only two congruences)
(c) $\mathcal{A}$ is a chain.

COROLLARY 1. *Let ϑ be some (possibly not proper) subvariety of ϑ_{LI}. Then ϑ is generated by all finite chains from ϑ.*

The following statement shows that the axiomatics of the variety ϑ_{LI} of all LI-algebras is adequate for fuzzy propositional logic.

COROLLARY 2. *The algebra $\mathcal{L}^+ = \langle [0,1], \vee, \wedge, \rightarrow, \neg, \sim \rangle$ generates the variety ϑ_{LI}.*

The finite chain $C_n \in \vartheta_{LI}$, where n is the cardinality of C_n, is said to be *even* if n is even, and *odd* otherwise.

Let us consider the variety $\vartheta^e_{LI} \subseteq \vartheta_{LI}$ generated by all even chains from ϑ_{LI}. ϑ^e_{LI} is a proper subvariety of ϑ_{LI}, since the equality $\ulcorner(a \rightarrow \sim a) \vee \ulcorner(\sim a \rightarrow a) = 1$ is valid in every even chain and false in every odd one.

THEOREM 5. *ϑ^e_{LI} is the only critical subvariety of ϑ_{LI}.*

References

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