Bulletin of the **S**ection of **L**ogic
Volume 6/1 (1977), pp. 15–23

reedition 2011 [original edition, pp. 15–26]

David Miller

ON DISTANCE FROM THE TRUTH AS A TRUE DISTANCE

This paper is a summary of a talk presented at the Fourth Scandinavian Logic Symposium, Jyväskylä, July 1976. A fuller version will appear in Acta Philosophica Fennica.

1. Popper's theory of verisimilitude ([13], pp. 228–237), or distance from the truth, was shown to be vacuous in [9] and [15]: it was proved that no false theory could be closer to the truth than was any other false theory. Indeed, amongst axiomatizable theories only true theories could be compared for verisimilitude ([9], p. 174).

The present paper tries a new approach to this problem – an approach that takes literally the idea of a theory's being distant from the truth. For if theories can be close to or distant from the truth, then presumably they can be close to or distant from one another. Accordingly we shall seek to define on pairs of theories a distance function $d(A, B)$ which satisfies the usual axioms for a metric operation: that is, $d(A, B)$ is zero if and only if $A = B$; $d(A, B) = d(B, A)$; and the triangle inequality must obtain. However, rather than make our distance function real valued, we shall require of it is an autometric function in the sense of [4]. That is to say, the values of the distance function will be elements of the very lattice of theories on which the function is defined.

2. We treat first the case of a Boolean algebra. Let us suppose that we have before us a finite language – by which I mean one in which all deductive theories are finitely axiomatizable. Then, factored by interderivability, these theories form a Boolean algebra $\mathcal{B}$. We shall call a function $* : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ autometric if

(1) $a * b = 0$ if and only if $a = b$
(2) $a * b = b * a$
(3) $a * c \leqslant (a * b) + (b * c)$.

Here $\leqslant$, 0, and + have their Boolean rather then their arithmetical meanings. The same holds, in what follows, for the symbols $\cdot$, $-$, and 1. The symmetric difference $a \triangle b$ of two elements is defined by

(4) $a \triangle b = a \cdot -b + -a \cdot b$.

THEOREM 1. (Ellis [4], Blumenthal [1]) *$a \triangle b$ is an autometric operation on $\mathcal{B}$.*

Following Ellis [5] we shall call a metric operation $*$ normal if there is an element e such that for every $a \in \mathcal{B}$

(5) $a * e = a$.

By (1), if there is such a base point it must be 0. The following result concerning normal autometric functions is adapted from the proof ([3], p. 325, [1], pp. 114f) that $\triangle$ is the only autometric function that is also a group operation.

THEOREM 2. *If $*$ is a normal autometric operation on $\mathcal{B}$ then*

$$a \triangle b \leqslant a * b \leqslant a + b.$$

Since all deductive theories of our language are finitely axiomatizable there is a sentence in it that expresses "the whole truth". In other words, there is an atom t of the algebra $\mathcal{B}$ that can be taken as representing "the truth". Distance from the truth will then be distance from t. And for every a, either $t \leqslant a$ (in which case we call a "true") or $t \leqslant -a$ in which case a is "false"). What Popper calls the truth content of a, namely the theory composed of all a's true consequences, is easily seen to be the element $a + t$.

THEOREM 3. *Let $*$ be a normal autometric operation on $\mathcal{B}$.*
If a is false then $a * t = a + t = a \triangle t$;
if a is true then $a * t = \begin{cases} a \\ a \cdot -t = a \triangle t \end{cases}$

We call an autometric operation $*$ downward strictly monotone if and only if

(6) $c < b < a \to b * a < c * a$,

and upward strictly monotone if and only if

(7) $c < b < a \to c * b < c * a$.

$*$ is strictly monotone if it satisfies both (6) and (7) (they are not by any means equivalent).

THEOREM 4. *If $*$ is a normal downward strictly monotone autometric operation on $\mathcal{B}$ then for every a we have $a * t = a \triangle t$.*

THEOREM 5. *If $*$ is a normal downward strictly monotone autometric operation on $\mathcal{B}$ then $b * t < a * t$ if and only if*
either $b < a$ and a and b have the same truth value
or $b \leqslant a + t$ and b is true and a is false.
Furthermore, $a * t = b * t$ if and only if $a = b$.

COROLLARY. *Under the conditions stated, if a is false then $(a+t)*t < a*t$.*

Thus the truth content of a false theory is closer to the truth than is the false theory itself. This is also a consequence of Popper's theory, and is surely desirable in any theory of verisimilitude.

3. We turn now to the general case, where it is not assumed that all theories are finitely axiomatizable. In particular, we shall suppose the theory T, the set of all true sentences, not to be axiomatizable, even recursively. Tarski showed that the ordering $\subseteq$ of set inclusion amongst deductive theories will generate a Heyting or pseudo-Boolean algebra. However, this lattice is inverted with respect to the sentential lattice from which it derives: S, the set of all sentences, corresponds to a contradiction, yet it is the widest element under the ordering $\subseteq$. It is therefore appropriate to consider the dual lattice $\mathcal{L}$, the lattice generated by $\supseteq$. In this section, accordingly, the ordering relation $\leqslant$ is identical with the superset relation $\supseteq$. The operation $-$ of subtraction exists in $\mathcal{L}$ (such lattices have been called subtractive in [2]; they are more usually known, following [8], as Brouwerian algebras): $B - A$ is defined as the logically strongest (that is, smallest under $\leqslant$) theory C for which $A \vee C$ is entailed by B (that is, $B \leqslant A \vee C$). We define the symmetric difference $A \triangle B$ by

(8) $A \triangle B = (A - B) \vee (B - A)$.

A function $* : \mathcal{L} * \mathcal{L} \to \mathcal{L}$ is called autometric if

(9) $A * B = S$ if and only if $A = B$
(10) $A * B = B * A$
(11) $A * C \leqslant (A * B) \vee (B * C)$.

THEOREM 6. (Nordhaus & Lapidus [12]) *$A \triangle B$ is an autometric operation on $\mathcal{L}$.*

An autometric operation $*$ is called normal if for all A we have $A * S = A$.

THEOREM 7. *If $*$ is a normal autometric operation on $\mathcal{L}$ then*

$$A \triangle B \leqslant A * B \leqslant A \vee B.$$

LEMMA. *$A \triangle T \neq A \vee T$ if and only if $A = C \vee T$ for some false theory C. In this case $A \triangle T = C$.*

THEOREM 8. *If $*$ is a normal autometric operation on $\mathcal{L}$ then $A * T = A \triangle T = A \vee T$ unless $A = C \vee T$ for some false theory C. In this case $A * T$ is equal to A or to C.*

THEOREM 9. *There is no autometric operation on $\mathcal{L}$ that is normal and downward strictly monotone.*

It follows from the lemma on p. 173 of [9] that if A is recursively axiomatizable then it is not truth content of any false theory. Thus for axiomatizable A we have $A * T = A \vee T$. The distance from the truth of an axiomatizable theory is equal to its truth content. Moreover, by the second corollary to Theorem 5 on the next page of [9], the truth content of an axiomatizable theory increases monotonically with its content.

THEOREM 10. *Suppose $*$ is a normal autometric operation on $\mathcal{L}$. Then if B and A are axiomatizable,*

$$B * T < A * T \text{ if and only if } B < A.$$

There is no straightforward way, within this framework, of avoiding the difficulties of Theorem 10: amongst axiomatizable theories approach to the truth is indeed independent of where the truth in truth is. But we can at least circumvent Theorem 9. We do this by representing each deductive theory A by the set $\mathcal{M}(A)$ of its "models", or equivalently by the set of all complete theories forms a Boolean algebra, of course; and on this we can impose a normal downward strictly monotone autometric $*$. The singleton $\mathcal{M}(T)$ is an atom in this algebra, so the results of the previous section tell us that $\mathcal{M}(A)*\mathcal{M}(T) = \mathcal{M}(A)\triangle\mathcal{M}(T)$, where $\triangle$ is here simply set-theoretical symmetric difference, or "disjoint union". Thus we can define the distance between A and T as $\mathcal{M}(A)\triangle\mathcal{M}(T)$. The conclusions of Theorem 5 apply, mutatis mutandis.

4. For the remainder of this paper we shall restrict ourselves to the study of distance functions in a Boolean algebra $\mathcal{A}$, which for ease of exposition we shall continue to think of as an algebra of theories. As just explained, this need not be regarded as a crippling restriction.

We derive easily from Theorems 3 and 5 that as long as $*$ is normal the stronger of two logically comparable false theories is close to the truth than is the weaker. Abstractly written, this is

(12) $b < a < -t \to b * t < a * t.$

Here t is assumed to be an atom. There are plenty of reasons for being unhappy with (12). Is it perhaps the assumption of normality that is responsible?

It turns out that we can weaken the normality assumption quite considerably, and still obtain (12). One way to do this is to require the operation $*$ to satisfy the following two conditions:

(13) $(a + b) * 0 = (a * 0) + (b * 0),$
(14) $-a * 0 = -(a * 0).$

That is to say, the singulary operation $*0$ is a homomorphism from $\mathcal{A}$ into $\mathcal{A}$. (Actually, we can say more than this, as is evident from Theorem 16 below.) Clearly every normal autometric operation $*$ satisfies (13) and (14).

THEOREM 11. *Let $*$ be an autometric operation on $\mathcal{A}$. Then if (13) and (14) hold, so does (12).*

The interest of conditions (13) and (14), and thus of Theorem 11, is that any homomorphism can be regarded as a "Boolean valued probability function", in the same way that $*$ is a "Boolean valued metric operation". Thus if (13) and (14) hold then $a * 0$ behaves formally like the probability of a. This is intuitively extremely satisfactory: what else is the probability of a but the distance of a from impossibility?

However, even (13) and (14) are unnecessarily strong for our purpose. We can replace (13) by the feebler ordering requirement

(15) $b \leqslant a \rightarrow b * 0 \leqslant a * 0,$

from which we can only derive

(16) $(a * 0) + (b * 0) \leqslant (a + b) * 0.$

Condition (15) is of course a consequence of what might be called the condition of upward weak monotony:

(17) $c \leqslant b \leqslant a \rightarrow c * b \leqslant c * a,$

together with the fact (2) that $*$ is a commutative operation. In passing, it is worth stating the following result.

THEOREM 12. *Let $*$ be an autometric operation on $\mathcal{A}$. Then (17) is satisfied if and only if the condition of downward weak monotony also holds:*

(18) $c \leqslant b \leqslant a \rightarrow b * a \leqslant c * a.$

This contrasts with the mutual independence of (6) and (7) noted above.

THEOREM 13. *Let $*$ be an autometric operation on $\mathcal{A}$. Then if (14) and (15) hold, so does (12).*

Now we are interested not only in the comparison of false theories with respect to distance from t, but also in the comparisons of false theories and their truth contents (as in the Corollary to Theorem 5). We certainly have

THEOREM 14. *Let $*$ be an autometric operation on $\mathcal{A}$. Then if (13) and (14) hold, so too does*

(19) $a \leqslant -t \rightarrow (a + t) * t \leqslant a * t.$

However, it is unlikely that this can be proved from (14) and (15). Further properties of the symmetric difference $\triangle$ that fail in general for autometric operations satisfying (13) and (14) are detailed in the next Theorem. In view of the fact that Theorem 3 did not specify the operation

$*$ quite uniquely, these results are hardly surprising.

THEOREM 15. *Let $*$ be an autometric operation on $\mathcal{A}$. Then the following are not consequences of (13) and (14).*

(20) $a \leqslant -t \to (a+t) * t < a * t$,
(21) $t \leqslant b < a \to b * t \leqslant a * t$.

It is obvious that (21) is an immediate consequence of (17).

Two further consequences of (14) and (15) are given in the next Theorem. The first (22) shows that $*0$ is not just a homomorphism from $\mathcal{A}$ into $\mathcal{A}$; it is actually a one-to-one mapping. For this we need to use the left-to-right implication of (1), which otherwise is only employed in the various proofs of (12).

THEOREM 16. *Let $*$ be an autometric operation on $\mathcal{A}$. Then if (14) and (15) hold, so too do*

(22) $b * 0 \leqslant a * 0 \to b \leqslant a$,
(23) $a * b = 0 \to a * b = (a * 0) + (b * 0)$.

As an application of (23), suppose we interpret complete theories, or atoms of $\mathcal{A}$, as "possible worlds". Then intuitively it seems reasonable that there should be four worlds, w, x, y, z, such that w and x are close together, and y and z are, but w and y (and x and z, and so on) are far apart. This, very surprisingly, is impossible if the distance function $*$ satisfies (14) and (15). For it follows from (23) that each of the four worlds is close to 0, so that $(w * 0) + (y * 0) = w * y$ is also small. That is, w is close to y.

5. This last argument can be made more compelling by going over to a numerical metric that accurately reflects the behaviour of the autometric $*$. Here I shall not consider numerical representations of $\triangle$ or $*$ except to note that it was already known to Frecht [6], Mazurkiewicz [7], and Nikodym [11] in the early thirties that the probability $p(a \triangle b)$ satisfies all the usual axioms for a pseudometric (see also Popper [14]). In the same way, $p(a * b)$ will be a pseudometric operation if $*$ is autometric. We cannot, however, except all of the above Theorems to survive the transformation of $\leqslant$ from a lattice ordering to a linear ordering.

A general theory of distance in a Boolean algebra is to be found in my paper [10].

References

[1] L. M. Blumenthal & K. Menger, **Studies in Geometry**, San Francisco (1970).

[2] H. B. Curry, **Foundations of Mathematical Logic**, New York (1963).

[3] J. G. Elliott, *Autometrization and the Symmetric Difference*, **Canadian Journal of Mathematics**, vol. 5 (1953), pp. 324–331.

[4] D. Ellis, *Autometrized Boolean Algebras I*, **Canadian Journal of Mathematics**, vol. 3 (1951), pp. 87–93.

[5] D. Ellis, *Notes on Abstract Distance Geometry I. The Algebraic Description of Ground Spaces*, **The Tôhoku Mathematical Journal**, ser. 2, vol. 3 (1951), pp. 270–272.

[6] M. Fréchet, *Nouvelles expressions de la "distance" de deux variables aléatoires et de la "distance" de deux fonctions measurables*, **Annales de la société polonaise de Mathématique**, vol. 9 (1930), pp. 45–49.

[7] S. Mazurkiewicz, *Ueber die Grundlagen der Wahrscheinlichtkeitarechnung I*, **Monatshefte für Mathematik und Physik**, vol. 41 (1933), pp. 343–352.

[8] J. C. C. McKinsey & A. Tarski, *On Closed Elements in Closure Algebras*, **Annals of Mathematics**, ser. 2, vol. 47 (1946), pp. 122–162.

[9] D. W. Miller, *Popper's Qualitative Theory of Verisimilitude*, **British Journal for the Philosophy of Science**, vol. 25 (1974), pp. 166–177.

[10] D. W. Miller, *New Axioms for Boolean Geometry*, **Bulletin of the Section of Logic** (to appear).

[11] O. Nikodym, *Sur une généralisation des intégrales de M. J. Radon*, **Fundamenta Mathematicae**, vol. 15 (1930), pp. 131–179.

[12] E. A. Nordhaus & L. Lapidus, *Brouwerian Geometry*, **Canadian Journal of Mathematics**, vol. 6 (1954), pp. 217–229.

[13] K. R. Popper, **Conjectures and Refutations**, London (1963).

[14] K. R. Popper, *A Note on Verisimilitude*, **British Journal for the Philosophy of Sciences**, vol. 27 (1976), pp. 147–159.

[15] P. Tichý, *On Popper's Definitions of Verisimilitude*, **British Journal for the Philosophy of Science**, vol. 25 (1974), pp. 155–160.

University of Warwick
England