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MATRICES FOR PREDICATE LOGICS

This is an abstract of a lecture read at the seminar of the Section of Logic, Polish Academy of Sciences, Wrocław, October 1976.

Let $\underline{F}$ be the set of all formulas built up by means of individual variables $x_1, x_2, \ldots$, predicate letters $P_1, P_2, \ldots$ (P_i being a $\lambda(i)$-ary predicate letter), connectives $F_1, \ldots, F_n$, and quantifiers $\forall$ and $\exists$, according to the usual definition of a formula.

By a (logical) *matrix* for $\underline{F}$ we shall mean any sequence $\underline{M} = (M, f_1, \ldots, f_n, \cap, \cup, B)$, where M is a non-empty set, $B \subseteq M$, $(M, f_1, \ldots, f_n)$ is an algebra similar to $(\underline{F}, F_1, \ldots, F_n)$, and both $\cap$ and $\cup$ are functions from 2^M into M.

By a *structure connected with* $\underline{M}$ (or simply a *structure* if $\underline{M}$ is fixed up) we mean any sequence $\underline{A} = (A, R_1, R_2, \ldots)$ where A is a non-empty set and for all $i \in \omega$, R_i is a function from $A^{\lambda(i)}$ into M. Let us denote $V = \{x_1, x_2, \ldots\}$. A *valuation* of $\underline{F}$ in $\underline{A}$ (where $\underline{A}$ is connected with $\underline{M}$) is any function φ from $V \cup \underline{F}$ into $A \cup M$ such that

$$\begin{aligned} \varphi(V) &\subseteq A \\ \varphi(P_i x_{j_i} \ldots x_{j_{\lambda(i)}}) &= R_i(\varphi x_{j_i}, \ldots, \varphi x_{j_{\lambda(i)}}) \\ \varphi(F_i \alpha_1 \ldots \alpha_k) &= f_i(\varphi\alpha_1, \ldots, \varphi\alpha_k) \\ \varphi(\forall x_i \alpha) &= \cap\{\varphi'\alpha : \varphi' =_i \varphi\} \\ \varphi(\exists x_i \alpha) &= \cup\{\varphi'\alpha : \varphi' =_i \varphi\} \end{aligned}$$

where "$\varphi' =_i \varphi$" stands for "$\varphi'(x_j) = \varphi(x_j)$ for all $j \neq i$". (Structure in a very similar sense are discussed e.g. in [1], [6].)

Let $V(\alpha)$ denote the set of all variables free in α, and let $\alpha[x_i/x_j]$ denote the result of substituting x_j for all free occurrences of x_i in α, with normal restrictions. If α, β are formulas ($\alpha, \beta \in \underline{F}$) then we shall say that α and β are *similar*, (in symbols $\alpha \sim \beta$), if one of them can be obtained from the other by changing some bound variables. A function $e : \underline{F} \to \underline{F}$ is said to be a *substitution* if e is a homomorphism with respect to $F_1, \ldots, F_n, \forall x_1, \forall x_2, \ldots, \exists x_1, \ldots$ and the following conditions are satisfied:

(a) $V(e\alpha) \subseteq V(\alpha)$, all $\alpha \in \underline{F}$

(b) for every atomic $\alpha \in \underline{F}$, and for every $i, j \in \omega$, there is some $\alpha' \in \underline{F}$ of a special form, see [3], such that $e\alpha \sim \alpha'$ and $e(\alpha[x_i/x_j]) \sim \alpha'[x_i/x_j]$.

(A detailed discussion of the notion of substitution in languages with quantifiers is contained in [3] and [4].)

Any pair $\underline{L} = (\underline{F}, C)$, where $C : 2^{\underline{F}} \to 2^{\underline{F}}$ satisfies the well-known Tarski's postulates for a consequence operation (cf. [5]), is called a *logic*. $\underline{L}$ is *finite* (and so is C) provided that $\alpha \in \bigcup\{C(Y) : Y \subseteq X \text{ and } \overline{\overline{Y}} < \aleph_0\}$ whenever $\alpha \in C(X)$. $\underline{L}$ is *structural* (and so is C) provided that $eC(X) \subseteq C(eX)$, for all $X \subseteq \underline{F}$ and for all substitutions e. $C_1 \leqslant C_2$ stands for "$C_1(X) \subseteq C_2(X)$, all $X \subseteq \underline{F}$".

Let $\underline{F}$ be the set of all formulas in some language, and let $\underline{M} = (M, f_1, \ldots, f_n, \cap, \cup, B)$ be a matrix for $\underline{F}$. Define operations $C_{\underline{M}}$, $C^*_{\underline{M}}$ and $\overline{C}_{\underline{M}}$ on $2^{\underline{F}}$ by

(A) $\alpha \in C_{\underline{M}}(X)$ iff for every structure $\underline{A}$ connected with $\underline{M}$ and for every valuation φ of $\underline{F}$ in $\underline{A}$, if $\varphi X \subseteq B$ then $\varphi\alpha \in B$ (comp. [2] for the case of sentential logics).

(B) $\alpha \in C^*_{\underline{M}}(X)$ iff for every structure $\underline{A}$ connected with $\underline{M}$ if for every valuation φ of $\underline{F}$ in $\underline{A}$, $\varphi X \subseteq B$, then for every valuation φ of $\underline{F}$ in $\underline{A}$, $\varphi\alpha \in B$.

(C) $\alpha \in \overline{C}_{\underline{M}}(X)$ iff $\varphi\alpha \in B$ for every valuation φ of $\underline{F}$ in any structure, whenever $\varphi X \subseteq B$ for every valuation φ of $\underline{F}$ in any structure.

One can easily check it that each one of the operations defined in (A), (B), (C) is a consequence in $\underline{F}$. Moreover

THEOREM 1. *For all* $\underline{M}$*:*

a. *Both* $C_{\underline{M}}$ *and* $C^*_{\underline{M}}$ *are structural.*
b. $\overline{C}_{\underline{M}}$ *is Post-complete.*
c. $C_{\underline{M}} \leqslant C^*_{\underline{M}} \leqslant \underline{C}_{\underline{M}}$.

(Note that $\overline{C}_{\underline{M}}$ is not in general a structural consequence operation.)

THEOREM 2. *If* $\underline{M}$ *is a finite matrix then* $C_{\underline{M}}$ $(C^*_{\underline{M}}, \overline{C}_{\underline{M}})$ *is a finite consequence operation.*

Now consider the following new notion of a matrix consequence. Let $\underline{M}$ and $\underline{F}$ be as above, and let h be a function from $\underline{F}$ into M. Then h is said to be a *valuation* of $\underline{F}$ in $\underline{M}$ provided that the following conditions hold:

$$hF_i\alpha_1 \dots \alpha_k = f_i(h\alpha_1, \dots, h\alpha_k)$$

$$h \overset{\exists}{\forall} x_i\alpha = \overset{\cup}{\cap} \{h\beta[x_i/x_j] : \beta \sim \alpha \text{ and } j \in \omega\}$$

Define $C^0_{\underline{M}}$ as follows:

(D) $\alpha \in C^0_{\underline{M}}(X)$ iff for every valuation h of $\underline{F}$ in $\underline{M}$, if $hX \subseteq B$ then $h\alpha \in B$.

PROBLEM. Prove (or disprove) that $C^0_{\underline{M}} = C_{\underline{M}}$.

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