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Giséle Fischer Servi

ON MODAL LOGICS WITH AN INTUITIONISTIC BASE
(Abstract)

In [6] Prior proposed a modal calculus, $MIPC$, which is an extension of the intuitionist propositional calculus. This system, which I call S^5IC, turns out to be the intuitionist analogue of Lewis's S^5 in the sense that

1) S^5IC plus $A \vee \neg A$ is equivalent to S^5,
2) collapsing the modal operators yields Heyting's calculus.

The question as to whether S^5IC is to be considered analogous to S^5 is not completely answered by 1) and 2) in so far that they are not sufficient for singling out the "correct" S^5-analogue. Bull however showed in [1] that S^5IC has another characteristic which definitely settles the question of analogy, i.e.

3) there is a translation map T, from S^5IC formulae into Intuitionistic predicate calculus (IPC) formulae with one variable for which

$$\vdash_{S^5IC} A \text{ iff } \vdash_{IPC} TA.$$

It is well known that 3) holds if (in both systems) we replace IC by the ordinary classical calculus. This third condition suggests that there might exist a general rule by which one could find, in correspondence to a classical modal system, its intuitionist counterpart. My purpose then is to define a general criterion for the concept of "intuitionist modal analogue" in the form of another theorem preserving translation which will be seen to be a natural generalization of Gödel's translation from IC to S^4.

The following considerations will be helpful in formulating my proposal. Let S^5IC be the above modal extension of the intuitionist propositional calculus with the usual connectives and operators $(\vee, \wedge, \rightarrow, \neg, L, M)$;

let $(S^4, S^5) - C$[1] be a "bimodal" calculus with a classical base $(\vee, \wedge, \rightarrow, \neg, L_1, L_2)$ and the following axioms and rules:

(i) S^4 – axioms and rules on L_1;
(ii) S^5 – axioms and rules on L_2;
(iii) suitable axioms relating L_1 to L_2;

Let T be a translation map from S^5IC to $(S^4, S^5) - C$ which extends the Gödel translation for IC in the following way:

$$T(MA) = \neg L_2 \neg TA$$
$$T(LA) = L_1 L_2 TA.$$

The idea is to prove the following:

THEOREM. $\vdash_{S^5IC} A$ *iff* $\vdash_{(S^4,S^5)-C} TA$

Now, let $*$ be a (classical) modal calculus of some sort (the class of such calculi must satisfy suitable conditions; e.g., $*$ must contain a full classical propositional base). Then it seems reasonable to assume the following definition for a $(* - IC)$:

DEFINITION. The axioms for $(* - IC)$ are those formulae whose T-translates are these of $(S^4, *)C$,
where $(S^4, *) - C$ is the obvious generalization of $(S^4, S^5) - C$. The first step in the program outlined is the study of the algebraic properties of S^5IC, $(S^4, S^5) - C$ and their relations. For this purpose I introduce the concept of *Monadic Heyting Algebras* (HM's) which are then proved to be equivalent to the *canonical models* defined in Bull [2]. The formulation in algebraic terms of the concepts and problems indicated above is then made possible and a few further results are proved.

References

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[1] A bimodal calculus which presents some similarities with $(S^4, S^5) - C$ was studied for its own sake as a plausible interpretation of a combined alethic-temporal modality, see Fitting [4].

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Via La Grola, 29
43100 Parma, Italia