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A CLASSIFICATION OF INCONSISTENT THEORIES

This paper sketches, in a very short way, some ideas connected with a suggestion how to define, in an appropriate way, the degree of consistency for inconsistent sets of formulas and how to define a classification of such sets. A detailed explanation with complete proofs can be found in the author's "Toward Many-Valued Logical Consistency" which has been submitted for publication in Studia Logica.

The request that any formalized theory should be consistent is one among the strictest in the methodology of science not admitting usually any discussion or compromise. The reason must be seen in the fact that in an inconsistent theory everything can be proved, i.e. such a theory cannot serve, neither partially, as a tool for segregating the true sentences from those, which are false.

However, some recent achievements of applied logic, namely in the field of artificial intelligence and robotics, pour a new light on this viewpoint. Formalized theories are built up automatically, by an automaton or a robot, on the ground of some observations and the used methods are far from being so sophisticated or "intelligent" as to ensure the consistency of the obtained theories. On the other hand, the necessity to solve, using these theories, in a real time some tasks in the environment excludes usually the possibility to apply some aposteriori consistency test, which is usually rather complex and expensive. The only we have to do is to utilize, somehow, even inconsistent theories supposing they are not "too much" inconsistent.

There are, al least, three criteria which should be taken into consideration having a set A of formulas: the minimal cardinality of an inconsistent subset $B \subset A$, the length of proof of its inconsistency and the complexity of a procedure which chooses B from A. The minimal length of proof of a fixed inconsistent formula from the formulas contained in B is denoted by $p(B)$, here length means the number of formulas or the total number of occurrences of symbols in the proof in question. A sequence $e(A) = \{e_1, e_2, \ldots\}$ is called an *enumeration* of A, if any $a_i \in A$ and there is, for any $x \in A$, $a_j \in N$ such that $x = a_j$, $N = \{0, 1, 2, \ldots\}$.

DEFINITION 1. Let A be a set of formulas, let $e(A)$ be one of its enumerations, let $k, m, n \in N$. The enumeration $e(A)$ is called $\langle k, m, n\rangle$ – *inconsistent*, if

$$p(\{e_k, e_{k+1}, \ldots, e_{k+m-1}\}) \leqslant n.$$

The set

$$S(e(A)) = \{\langle k, m, n\rangle : p(\{e_k, e_{k+1}, \ldots, e_{k+m-1}\}) \leqslant n\}$$

will be called *set-valued degree of inconsistency of the enumeration* $e(A)$.

Some other criteria of the degree of inconsistency can be expressed using the values $S(e(A))$.

(a) A set A of formulas is consistent in the usual sense iff for any enumeration $e(A)$ of A the set $S(e(A))$ is empty.

(b) An enumeration $e(A)$ is called simple, if there is, in case A is finite, just one occurrence of any formula from A among $\{e_1, e_2, \ldots, e_{card(A)}\}$ or if there is, in case A is not finite, just one occurrence of any formula from A among $\{e_1, e_2, \ldots\}$. Then the maximal cardinality of a consistent subset of A can be expressed as

$$max\{max\{i : (\{1\} \times N_i \times N) \cap S(e(A)) = \emptyset\}\},$$

where $N_i = \{1, 2, \ldots, i\}$ and the first maximum is taken over the set of all simple enumeration of A.

(c) The minimal cardinality of an inconsistent subset of A can be expressed as

$$min\{min\{j : (\{1\} \times N_j \times N)\ S(e(A)) \neq \emptyset\}\}$$

with the first minimum taken over the set of all enumerations of A.

(d) The minimal possible length of a proof of the fixed inconsistent formula (see the Parikh's criterion used in [3], [4]) can be expressed in the form

$$min\{j : (N \times N \times N_j) \cap S(e(A)) \neq \emptyset\}.$$

(e) The set A of formulas is called $\langle r, s\rangle$ – locally consistent, $r, s \in N$, if there is no subset $A_0 \subset A$ such that $card(A_0) \leqslant r$ and $p(A_0) \leqslant s$. This fact can be expressed as

$$\bigcup_{e(A)} S(e(A)) \subset N \times (N - N_r) \times (N - N_s).$$

Let us remark that in the author's paper mentioned above the notion of set-valued degree of inconsistency is defined in a slightly generalized way taking into consideration also an a priori given importance or weight of various formulas from A, a simplified form is used here only because of a limited extend of this paper.

Even if the set-valued degree of inconsistency is a very detailed source of information concerning the specific features of an inconsistent set of formulas, the limited possibilities of manipulation with these values makes it desirable to map them, in an appropriate way, into the real line. The three following possibilities will be suggested.

Let $E \subset N \times N$, $E = \bigcup E_i$, $E_i = \{\langle k, l\rangle : k \geqslant i_1, l \geqslant i_2\}$, $i, i_1, i_2, k, l \in N$. Set

$$d_i(E) = min\{j : j \in N,\ \langle i, j\rangle \in E\}$$

if such a j exists, $d_i(E) = 0$ otherwise. As can be easily seen, the sequence $\{d_1(E), d_2(E), \ldots\}$ defines unambiguously the set E, moreover, there is an index $i_0(E) \in N$ such that $d_i(E) = d_{i_0(E)}(E)$ for any $i \geqslant i_0$. Hence, denoting by $Pr(i)$ the i-the prime in the usual increasing order, the number

$$s(E) = \Pi_{i=1}^{i_0(E)} (Pr(i))^{d_i(E)}$$

defines unambiguously the set E. For any enumeration $e(A)$ of a set of formulas the set $S(e(A)) \subset N^3$ can be considered as a sequence $S_n(e(A))$ of sets from $N \times N$ of the same type as E above, so we have:

Definition 2. An enumeration $e(A)$ of a set A of formulas is called *recursively inconsistent*, if the function

$$f(n) = f(n, e(A)) = s(S_n(e(A)))$$

is recursive. The inverse value of the Gödel number of the function $f(., e(A))$ is called the *absolute numerical degree of inconsistency* of the enumeration $e(A)$ and denoted by $s(e(A))$.

THEOREM 1. *The value $s(e(A))$ defines unambiguously the set $S(e(A))$.*

This theorem justifies the adjective "absolute" in Definition 2. However, the significance of this criterion is almost purely of theoretical nature because it is extremely difficult to compute the values of $s(e(A))$ and to operate with them. That is why the following more rough criteria are of importance.

DEFINITION 3. A function f defined in the field of all subsets of N^3 and taking its values in $\langle 0, 1\rangle$ is called *monotonous numerical degree of inconsistency*, if for any two possible values A, B of the set-valued degree of inconsistency $A \subset B$ implies $f(A) \leqslant f(B)$.

THEOREM 2. *Let $\rho : N^3 \times N^3 \to \langle 0, \infty)$ be a metric on N^3. If $a \in N^3$, $A \subset N^3$, set*

$$\rho(a, A) = min\{\rho(a, x) : x \in A, \rho(a, \emptyset) = \infty\}.$$

Then the function

$$f(S(e(A))) = (\rho(\langle 0, 0, 0\rangle, S(e(A))))^{-1}$$

is a monotonous numerical degree of inconsistency.

Let $\mathcal{F}$ be a σ-field of subsets of N^3, let μ be a measure defined on $\mathcal{F}$ so that $\langle N^3, \mathcal{F}, \mu\rangle$ is a space with measure. Set, for any $A \subset N^3$

$$\mu^*(A) = inf\{\mu(B) : B \supset A, B \in \mathcal{F}\},$$

$$\mu_*(A) = sup\{\mu(B) : B \subset A, B \in \mathcal{F}\},$$

(the outer and the inner measure associated with μ, cf. [1]).

THEOREM 3. *Let $\langle N^3, \mathcal{F}, \mu\rangle$ be a space with measure. Both the functions*

$$f_1(S(e(A))) = \mu^*(S(e(A))),$$

$$f_2(S(e(A))) = \mu_*(S(e(A)))$$

are monotonous numerical degrees of inconsistency.

The foregoing theorems show a general way in which a rather large class of numerical degrees of inconsistency can be obtained starting from the set-valued degree of inconsistency S as the basis of our attempts to describe and handle somehow the inconsistent sets of formulas. Even the criteria (a) – (e) mentioned above can be expressed in this way, e.g. taking $\mathcal{F} = \mathcal{P}(N^3), \mu(\emptyset) = 0$, $\mu(A) = 1$, $A \neq \emptyset$, we have immediately $f_1(S(e(A))) = f_2(S(e(A))) = 0$ iff A is consistent, this value being 1 in the opposite case. As another example let us take the criterion of local consistency. The function

$$\rho(\langle k, m, n\rangle, \langle k', m', n'\rangle) = \begin{array}{l} max\{|m - m'| - r, 0\}+ \\ +max\{|n - n'| - s, 0\} \end{array}$$

can be easily proved to be a metric and a set A of formulas is $\langle r, s\rangle$-locally consistent iff

$$\rho(\langle 0, 0, 0\rangle, S(e(A))) > 0.$$

References

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