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THE CRITERION OF BROUWERIAN AND CLOSURE ALGEBRAS TO BE FINITELY GENERATED

When investigating the semantics of superprintuitionistic logics and extensions of the modal system $S4$, as well as from purely algebraic point of view, it is important to know characteristic properties and structure of finitely generated Brouwerian (alias pseudo-Boolean) algebras [1] and closure algebras [1].

In this paper we propose a criterion for the algebras to be finitely generated.

The duality between closure algebras (resp., Brouwerian lattices) and perfect Kripke models, on which this paper rests, is developed in [2]. The key definitions are recalled here for convenience.

A subset A of a pre-ordered set (X, R) is a cone if $x \in A$ and xRy imply $y \in A$.

The following proposition will be useful.

Proposition 1. *Let (X, R) be a pre-ordered set; let E be a partition of X. Then the following conditions are equivalent:*

(1) If $A \subseteq X$ is a cone, then the saturation of A with respect to E, i.e. the set $E(A) = \bigcup\{E(x) : x \in A\}$, is a cone;

(2) If A is a set saturated relative to E (i.e. $E(A) = A$), then $R^{-1}A$ is saturated set too.

Recall that a Kripke model (X, R), i.e. a non-empty set X ("worlds") with a pre-ordered set relation R ("accessibility relation") is said to be *perfect* [2], if X is a 0-dimensional Hausdorff compact space (a Stone space), the sets $R(x) = \{y : xRy\}$ are closed for any $x \in X$ and $R^{-1}clA = clR^{-1}A$ for any $A \subseteq X$, where cl is a closure operation of the topological space

X. Let Cat_1 be the category of perfect Kripke models and Cat_2 its full subcategory, in which R is an order relation. It was shown in [2] that Cal (resp., Brouw) is dual (under a contra-variant functor) to the category Cat_1 (resp., Cat_2).

More explicitely, if $(B, c) \in Cal$, then (B, c) can be identified with the lattice of clopen subsets of its dual model (X, R). Let $(X, R) \in Cat_1$. A partition E of the set X is said to be correct [2] if the classes $E(x)$ $(x \in X)$ of this partition are compact sets and the saturation (with respect to E) of any cone is a cone.

PROPOSITION 2 [2]. *There is a one-to-one correspondence between subalgebras of the algebra $(B, c) \in Cal$ and correct partitions of the model $(X, R) = (B, c)^*$ (where $(B, c)^*$ denotes the object of the category Cat_1 which is dual to the object $(B, c) \in Cal$). Moreover, proper subalgebras correspond to non-trivial partitions (the partition E is non-trivial if $(\exists x \in X)(E(x) \neq \{x\})$.*

Let $(B, c) \in Cal$ and $(X, R) = (B, c)^*$. Let (B_0, c_0) be a subalgebra of (B_0, c_0) and let E_0 be the partition (corresponding to of the model (X, R). We have

PROPOSITION 3. *The following conditions are equivalent:*

(1) an element a of B belongs to B_0 (i.e. $a \in B_0$);

(2) the clopen $A = a^*$ (corresponding to $a \in B$) is "cut" by no class of partition E, i.e.

$$(\forall x \in X)(A \cap E(x) \neq \emptyset \Rightarrow E(x) \subseteq A);$$

(3) clopen A is saturated with respect to E, i.e. $E(A) = A$.

1. Coloration of models

Let (B, c) be a closure algebra and let $g_1, \ldots, g_k \in B$ be fixed elements. Let $(X, R) = (B, c)^*$ and let $G_i = g_i^*$ be the clopen corresponding to an element g_i $(1 \leqslant i \leqslant k)$. We ascribe an index p to a point $x \in X$ or, figuratively speaking, colour the point $x \in X$ with the colour p $(0 \leqslant p < 2^k)$, if $x \in G_1^{\delta_1} \cap \ldots \cap G_k^{\delta_k}$, where the sequence $(\delta_1, \ldots, \delta_k)$ is the dyadic entry of the number p, $G_i^{\delta_i} = X - G_i$, if $\delta_i = 0$ and $G_i^{\delta_i} = G$, if $\delta_i = 1$. We shall denote the set $G_1^{\delta_1} \cap \ldots G_k^{\delta_k}$ by X_p for brevity. It is clear that if $p \neq q$, then $X_p \cap X_q = \emptyset$ and $\bigcup\{X_p : 0 \leqslant p < 2^k\} = X$. The family $\{X_p : 0 \leqslant p < 2^k\}$ will be called coloration induced by elements $g_1, \ldots, g_k \in B$.

LEMMA. *Let $(X, R) \in Cat_1$, $G_i \subseteq X$ $(1 \leqslant i \leqslant k)$, and let E be a correct partition of the set X. The following conditions are equivalent:*

(1) the sets G_i are saturated with respect to E, i.e. $E(G_i) = G_i$ $(1 \leqslant i \leqslant k)$

(2) the classes X_p $(0 \leqslant p < 2^k)$ are saturated with respect to E, i.e. $E(X_p) = X_p$.

PROOF. Let $\underline{F} = \underline{F}(G_1, \ldots, G_k)$ be the smallest field of subsets of X containing all G_i $(1 \leqslant i \leqslant k)$. It is clear that $X_p \in \underline{F}$ and every (non-empty) X_p is an atom of Boolean algebra $\underline{F}$. Any $A \in \underline{F}$ is a union of atoms. It is clear that if atoms X_p are saturated, then any $A \in \underline{F}$, and hence any G_i $(1 \leqslant i \leqslant k)$ are saturated. In order to show the converse it is sufficient to notice that the operations $\cup, \cap, -$ preserve the property of saturating.

2. Criterion

Let $(B, c) \in Cal$, let $g_1, \ldots, g_k \in B$ be fixed elements, and $(X, R) = (B, c)^* \in Cat_1$. Let $G_i \subseteq X$, where $G_i = g_i$ and $\{X_p : 0 \leqslant p < 2^k\}$ is the criterion of the model (X, R) induced by sets G_i $(1 \leqslant i \leqslant k)$.

GENERATION THEOREM. *The following assertions are equivalent:*

(1) The closure algebra (B, c) is generated by the elements $g_1, \ldots, g_k \in B$, i.e. coincide with the smallest subalgebra containing the elements $g_1, \ldots, g_k$,

(2) Any non-trivial correct partition E of the model (X, R) contains many-coloured class of the partition E.

PROOF. (1) $\Rightarrow$ (2). Let (B, c) be generated by the elements $g_1, \ldots, g_k$ and let E be any non-trivial correct partition of the model (X, R). Then the subalgebra (B_0, c_0) of the algebra (B, c), corresponding to the partition E (Proposition 2), is proper and therefore there is a generator g_i such that $g_i \notin B_0$. Using Proposition 3, we have $E(G_i) \neq G_i$ and by Lemma $E(X_p) \neq X_p$ for some p; therefore is a non-trivial class $E(x)$ $(x \in X)$ of the partition E such that $E(x) \cap X_p \neq \emptyset$ and $E(x) \cap \overline{X_p} \neq \emptyset$, i.e. $E(x)$ is a many-coloured class.

(2) $\Rightarrow$ (1). Suppose that (B, c) is not generated by the elements $g_1, \ldots, g_k \in B$, i.e. that the smallest subalgebra (B_0, c_0) containing $g_1, \ldots, g_k$ is a proper subalgebra of (B, c). Then the correct partition E corresponding to (B_0, c_0) is not trivial. Let us show that any class of the partition E contains elements of the same colour. In fact, the partition class having been found, say $E(x)$, we should have $E(X_p) \neq X_p$, where p is a colour of the element x. But then, by Lemma, $E(G_i) \neq G_i$ for suitable i and, by Proposition 3, $g_i \notin B_0$ which contradicts the supposition.

Remark. If in the theorem "Brouwerian algebra T" is substituted for "closure algebra (B, c)", then the proposition preserves its validity.

Let (X, R) be a perfect Kripke model and let I be a non-empty finite set. The partition $\{X_p : p \in I\}$ of the set X whose every class X_p is compact, will be called a coloration.

The Colouring Theorem. *A closure algebra (B, c) is finitely generated iff the model $(X, R) = (B, c)^*$ permits such a coloration $\{X_p : p \in I\}$ that every non-trivial correct partition E of the model (X, R) has a many-coloured class.*

Application. Now we shall use our theorem for analysis of cyclic algebras. Recall that an algebra $(B, c) \in Cal$ is said to by cyclic if there is an element $g \in B$ generating (B, c). In this case it is clear that the coloration induced by g is bicoloured; for clarity we shall choose white $(p = 0)$ and black $(p = 1)$ colours.

(I). Let (B, c) be a "linear" closure algebra, i.e. an algebra such that its Kripke model $(X, R) = (B, c)^*$ is linear. Let $X = \{1, 2, 3, \ldots, n\}$ and let R be the natural order. Let G be the set of odd numbers of X. Let us show, using the criterion, that $g = G^*$ generates (B, c).

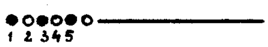

Fig.1.

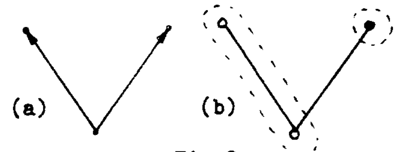

Fig.2

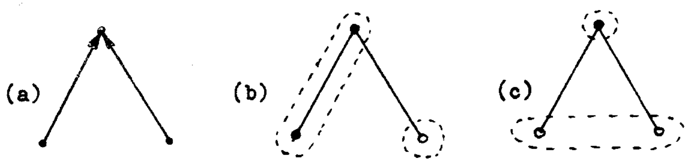

Fig.3.

The coloration induced by the set G is indicated in Fig. 1 (white and black points). It is clear (see Fig. 1) that every non-trivial correct partition E of X has a many-coloured class.

COROLLARY. *Any finite "linear" closure algebra (B, c) is cyclic. Emphasize that there is (up to isomorphism) only three cyclic linear Brouwerian algebras.*

(II). Fig. 2 presents the Kripke model consisting of theree points and the coloration of the model with necessary (by Theorem) properties. It reveals also that the corresponding closure algebra and Brouwerian algebra are cyclic.

On the other hand, closure algebras and Brouwerian algebras corresponding to the "turned-over" model (Fig. 3) are not cyclic, since no coloration (two main colorations are shown on Fig. 3b and 3c) satisfies the requirements of the theorem.

(III). Using criterion, we can easily check that infinite free cyclic closure algebras of a number of varieties (described in [3]) are cyclic.

(IV). Using colouring theorem it is easy to see that if an algebra $(B, c) \in Cal$ is finitely generated, then the corresponding model $(X, R) = (B, c)^*$ has the following two properties: (1) the set $MaxX$ of maximal elements of the model (X, R) is finite and (2) every clot (i.e. every set of the kind $R(x) \cap R^{-1}(x)$, $x \in X$) is finite.

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