Bulletin of the **S**ection of **L**ogic
Volume 6/2 (1977), pp. 53–60

reedition 2011 [original edition, pp. 53–63]

David Miller

NEW AXIOMS FOR BOOLEAN GEOMETRY

1. It was known already in the early thirties (for references see the next to last paragraph of my [3]) that if a Boolean algebra $\mathcal{A}$ is endowed with a measure μ, then the measure $\mu(a \triangle b)$ of the symmetric difference $a \triangle b$ of two elements satisfies all the standard axioms for a pseudometric operation. (If μ is strictly positive then we have a genuine metric operation.) That the symmetric difference itself formally satisfies the metric axioms, provided we give $\leqslant, +$, and 0 Boolean rather than arithmetic interpretations was first announced by Ellis [2]; and Blumenthal [1] noted that we can in fact metrize an arbitrary set by means of a Boolean valued (rather than real valued) function. Such structures he called Boolean metric spaces, or, more picturesquely, Boolean geometries. It is my intention, however, to use the latter term exclusively for the case where Boolean algebras are metrized by Boolean algebras. A Boolean geometry, therefore, is a triple $\langle \mathcal{A}, \mathcal{B}, * \rangle$ where $\mathcal{A}$ and $\mathcal{B}$ are Boolean algebras and $* : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ is a pseudometric operation satisfying axioms that have yet to be stated.

The philosophical interest of Boolean geometry is illustrated in [3]. The main problem is that of 'distance from the truth', where 'the truth' can be taken as an atom t in the algebra $\mathcal{A}$. I showed in Theorem 4 of [3] that an operation $* : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ that satisfies the usual metric axioms

(α) $a * b = 0 \leftrightarrow a = b$
(β) $a * b = b * a$
(γ) $a * c \leqslant (a * b) + (b * c)$,

together with the special conditions of normality (δ) and downward strict monotony (ε),

(δ) $a * 0 = a$

(ε) $c < b < a \to b * a < c * a$,

is uniquely determined with regard to distance from the truth t:

(ξ) $a * t = a \triangle t$,

where $a \triangle b$ is the symmetric difference of a and b. The uniqueness proof is in several respects unsatisfying. This is partly because of (δ), a condition which may not even be applicable if the algebra of values $\mathcal{B}$ is not the same as the metrized algebra $\mathcal{A}$; we can however replace (δ) by the two conditions

(η) $(a + b) * 0 = (a * 0) + (b * 0)$
(ϑ) $-a * 0 = -(a * 0)$,

which together state that $*0$ is a homomorphism, in order to overcome this difficulty. It is partly because one half of condition (α), namely

(ι) $a * b = 0 \to a = b$,

is not used. But it is mainly an account of condition (ε), which is patently introduced ad hoc in [3] simply in order to disqualify other possible candidates for the function $a * t$. I noted there that the natural companion to (ε), the condition of upward strict monotony

(κ) $c < b < a \to c * b < c * a$

is not equivalent to (ε), even though the corresponding two conditions of weak monotony

(λ) $c \leqslant b \leqslant a \to b * a \leqslant c * a$,
(μ) $c \leqslant b \leqslant a \to c * b \leqslant c * a$,

are interderivable, in the presence of the metric axioms (α), (β), (γ). More generally, by trafficking in strict inequalities (ε) renders itself algebraically a bit of a nuisance (just as (ι) is). It would be pleasant indeed if it could be dropped in favour of (λ) or (μ).

It is natural to try to generalize (ξ) first to

(ν) $a * b = a \triangle b$,

and then, in view of (η) and (ϑ), to

(ξ) $a * b = (a \triangle b) * 0$.

It is the purpose of this paper to provide axioms for $*$ which do indeed have equation (ξ) as a consequence. Thsese axioms will be axioms for a pseudometric only; that is, they will satisfy (β) and (γ), together with the weakened form of (α)

(o) $a * a = 0$.

The formulas (η), (ϑ), (λ), (μ) will be theorems of the system. In addition, (ε) and (κ) will be interderivable with each other and with the unretained half of (α), namely (ι). The axioms, in contrast to (α), (ε), (κ), (λ), (μ), (ι), will all be expressed as functional inequalities for the operation $*$.

2. The axioms of the system are three:

(1) $a * c \leqslant (b * a) + (b * c)$
(2) $(b + a) * (b + c) \leqslant a * c$
(3) $-a * c = -(a * c)$.

Axiom (1) is a form of the triangle inequality (γ); axiom (2) concerns the distance between Boolean sums, and was introduced in the first instance chiefly as a means of proving (λ) and (μ) (though its force is much more than this); and (3) is a natural generalization of (γ).

We first prove (o), (β), and (γ), the usual pseudometric axioms.

(4)	$a * a \leqslant b * a$	(1)
(5)	$a * a \leqslant -a * a$	(4)
(6)	$a * a \leqslant -(a * a)$	(3), (5)
(7)	$a * a = 0$	(6)
(8)	$a * b \leqslant (b * a) + (b * b)$	(1)
(9)	$a * b \leqslant b * a$	(7), (8)
(10)	$a * b = b * a$	(9)
(11)	$a * c \leqslant (a * b) + (b * c)$	(1), (10)

Thus every function $*$ satisfying (1)-(3) is a pseudometric. In what follows we shall not bother to cite uses of (10), the commutativity of $*$. It is enough to note that it depends on (1) and (3) only.

THEOREM 1. *The axioms (1), (2), (3) are completely independent in the sense of Moore [4].*

THEOREM 2. *The sentence* (ι) *is not derivable from (1), (2), (3).*

We now prove (λ) and (μ).

(12)	$c \leqslant b \leqslant a \to a * b \leqslant a * c$	(2)
(13)	$c \leqslant b \leqslant a \to b * a \leqslant c * a$	(12)
(14)	$c \leqslant b \leqslant a \to (a * b) + (a * c) = a * c$	(12)
(15)	$c \leqslant b \leqslant a \to b * c \leqslant a * c$	(1), (14)
(16)	$c \leqslant b \leqslant a \to c * b \leqslant c * a$	(15)

Axiom (2) has a dual of some interest:

(17)	$(-b + -a) * (-b + -c) \leqslant -a * -c$	(2)
(18)	$-(b.a) * -(b.c) \leqslant -a * -c$	(17)
(19)	$b.a * b.c \leqslant a * c$	(3), (18)

The next set of derivations, which are of crucial importance, are concerned with the lattice points in the Boolean algebra $\mathcal{A}$. That is to say, we are interested in the quadrangles formed by any two points a and b and their join and meet $a + b$ and $a.b$ respectively. We shall show that these quadrangles have all the properties of Euclidean rectangles: their opposite sides are equal (24), and their diagonals are equal (34).

(20)	$(a + b) * (a + b.a) \leqslant b * b.a$	(2)
(21)	$(a + b) * a \leqslant b * b.a$	(20)
(22)	$b.(a + b) * b.a \leqslant (a + b) * a$	(19)
(23)	$b * b.a \leqslant (a + b) * a$	(22)
(24)	$(a + b) * a = b * b.a$	(21), (23)
(25)	$(a + b) * c \leqslant ((a + b) * b) + (b * c)$	(1)
(26)	$c \leqslant b \to (a + b) * c \leqslant ((a + b) * (c + b)) + (b * c)$	(25)
(27)	$c \leqslant b \to (a + b) * c \leqslant (a * c) + (b * c)$	(26)
(28)	$c \leqslant b \to b * c \leqslant (a + b) * c$	(16)
(29)	$c \leqslant a \;\&\; c \leqslant b \to (a + b) * c = (a * c) + (b * c)$	(27), (28)
(30)	$(a + b) * a.b = (a * a.b) + (b * a.b)$	(29)
(31)	$a * b \leqslant (a * a.b) + (b * a.b)$	(1)
(32)	$a.a * a.b \leqslant a * b$	(19)
(33)	$a * b = (a * a.b) + (b * a.b)$	(31), (32)
(34)	$a * b = (a + b) * a.b$	(30), (33)

The proof of (ξ) now follows quickly.

(35)	$-a * b = (-a + b) * -a.b$	(34)
(36)	$-a * b = -(a. - b) * -a.b$	(35)
(37)	$a * b = a. - b * -a.b$	(3), (36)
(38)	$a * b = (a \triangle b) * 0$	(34), (37)

Thus the values of the function $*$ are determined by the special cases of the values of the function $*0$. Note, moreover, that the homomorphism conditions (η) and (ϑ) are immediate consequences of (29) and (3):

(39)	$(a + b) * 0 = (a + 0) * (b + 0)$	(29)
(40	$-a * 0 = -(a * 0)$	(3)

I have suggested in [3] that the function $*0$ can be regarded as a Boolean valued probability function. If this interpretation is accepted we shall say that all distance in $\mathcal{A}$ are determined by probability values (and, of course, vice versa). The following Theorem sums up the matter formally.

THEOREM 3. *The theories (1), (2), (3) and (38), (39), (40) are interderivable.*

If $\mathcal{A}$ is finite it is also of course the case that all distances are fixed by the probabilities of the atoms. What is more, they are also determined uniquely by the distances between the atoms, except in the case where $\mathcal{A}$ has only two atoms x, $-x$; for here, by (3) and (7), the distance between the atoms is compelled to be 1, and has no bearing on the values of $x * 0$ and $-x * 0$.

3. In this section we discuss matters relating to zero distance. We start by proving a few simple results.

(41)	$a * b = 0 \to a * c \leqslant b * c$	(1)
(42)	$a * b = 0 \to a * c = b * c$	(41)
(43)	$(a + d) * c \leqslant ((b + d) * c) + ((a + d) * (b + d))$	(1)
(44)	$(a + d) * c \leqslant ((b + d) * c) + (a * b)$	(2), (43)
(45)	$a * b = 0 \to (a + d) * c = (b + d) * c$	(44)
(46)	$c * d = 0 \to (a + c) * (a + d) = 0$	(2)
(47)	$a * b = 0\ \&\ c * d = 0 \to (a + c) * (b + d) = 0$	(1), (46)
(48)	$a * b = 0\ \&\ c * d = 0 \to a * c = b * d$	(42)

It follows easily from lines (3), (7), and (47) that

THEOREM 4. *The relation $a * b = 0$ is a congruence relation on $\mathcal{A}$.*

Indeed, something rather stronger obtains. We define terms in the usual way: all variables are terms, and the constants 0 and 1 are terms; if u and v are terms, so are $-u$, $u + v$, $u * v$; these are all the terms. We write $E(u, v)$ for the equation linking the terms u and v, and $\models E(u, v)$ to mean that the equation holds. The term u^s/t is like u except perhaps for containing the term t at some places where u contains the term s.

THEOREM 5. *If $\models E(s * t, 0)$ then $\models E(u * v, u^s/t * v^s/t)$.*

Line (42) tells us that if the distance between two points is 0 then they are equidistant from every point. There is a strong converse to this:

(49)	$a * c = b * c \to a * b \leqslant a * c$	(1)
(50)	$a * c = b * c \to a * -c = b * -c$	(3)
(51)	$a * c = b * c \to a * b \leqslant a * -c$	(49), (50)
(52)	$a * c = b * c \to a * b \leqslant (a * c).(a * -c)$	(49), (51)
(53)	$a * c = b * c \to a * b = 0$	(3), (52)

We turn now to the connection between the conditions (ε) and (κ) of strict monotony and the condition (ι) which turns the function $*$ from a pseudometric to a real metric.

THEOREM 6. *The sentences (ε) and (κ) are interderivable in the theory (1), (2), (3).*

PROOF. Suppose that (ε) holds, and that $c < b < a$. Then $-a < -b < -c$, so that by (ε) we have $-b * -c < -a * -c$. The result follows by (3) and (10).

THEOREM 7. *The sentences (κ) and (ι) are interderivable in the theory (1), (2), (3).*

PROOF. That (ι) entails (κ) is an immediate consequence of (53) and (10), given that (μ) is a theorem of our system (namely, (16)). For the converse, suppose that (κ) holds and that a is distinct from b. Then at least one of them, say a, is distinct from $a + b$. We can assume, therefore, that $0 \leqslant a < a + b \leqslant 1$. There are three possibilities:

(i) $0 = a < a + b = 1$. Thus $a * (a + b) = 0 * 1 = 1$. It follows that $a * (a + b) \neq 0$.

(ii) $0 < a < a + b$. By (κ), $0 * a < 0 * (a + b)$. By (42), $a * (a + b) \neq 0$.

(iii) $a < a + b < 1$. This time we use (ε) to infer that $a * 1 < (a + b) * 1$. Again by (42) we have $a * (a + b) \neq 0$.

In every case $a * (a + b) \neq 0$. But by (2), $(a + a) * (a + b) \leqslant a * b$. It follows that $a * b > 0$.

4. In this final section I shall discuss very briefly other ways of presenting the theory of distance in Boolean algebras.

Note first the following result, which shows that the symmetric difference is in effect the minimal metric that satisfies (1), (2), (3):

(54) $a \triangle b \leqslant c \triangle d \to a * b \leqslant c * d$ (16), (38)

Since, as is easily checked, $(b + a) \triangle (b + c) \leqslant b \triangle (a + c)$ is a valid inequality of Boolean algebra, we have as a further result of our system:

(55) $(b + a) * (b + c) \leqslant b * (a + c)$ (54)

This consequence is very similar to axiom (2). We can see, for example, that (13) follows simply, since we obtain at once

(56) $b \leqslant a \leqslant c \to a * c \leqslant b * c$ (55)

In fact, we can also prove (34), and therefore (38), and therefore (2), using (1), (3), and (55) only.

(57)	$b.a * b.c \leqslant b * a.c$	(3), (55)
(58)	$c \leqslant b \to (a + b) * c \leqslant (b * (a + c)) + (b * c)$	(26), (55)
(59)	$c \leqslant b \to b * c \leqslant (a + b) * c$	(1), (56)
(60)	$c \leqslant b \to b * (a + c) = (b + c) * (a + c) \leqslant c * (a + b)$	(55)
(61)	$c \leqslant b \to (a + b) * c = (b * (a + c)) + (b * c)$	(58), (59), (60)
(62)	$(a + b) * a.b = (b * a) + (b * a.b)$	(61)
(63)	$b * a.b = b.1 * b.a \leqslant b * 1.a = b * a$	(57)
(64)	$(a + b) * a.b = a * b$	(62), (63)

Theorem 8. *The theories (1), (2, 93), and (1), (55), (3) are interderivable.*

The theory outlined above does have a numerical version, which can be given by the following axioms for a real valued function $d(a, b)$ on $\mathcal{A} * \mathcal{A}$.

(π) $d(a, c) \leqslant d(b, a) + d(b, c)$
(ϱ) $d(b + a, b + c) + d(b.a, b.c) \leqslant d(a, c)$
(σ) $d(a, b) + d(-a, b) = d(-b, b)$

It will be observed that (ϱ) is the analogue not so much of (2) as of (2) and (19) together. This is because $a \leqslant c \ \& \ b \leqslant c \rightarrow a + b \leqslant c$ is a truth of Boolean algebra that has no parallel in the real numbers. The most important consequence of (π), (ϱ), and (σ) are

(τ) $d(0, 0) = 0$
(υ) $d(a, 0) + d(b, 0) = d(a + b, 0) + d(a.b, 0)$
(φ) $d(a, b) = d(a \triangle b, 0)$.

THEOREM 9. *The theory* $(\pi), (\rho), (\sigma)$ *is equivalent to the Kolmogorov axioms for probability shorn of the normalization condition* $p(\Omega) = 1$.

References

[1] L. M. Blumenthal & K. Menger, **Studies in Geometry**, San Francisco, 1970.

[2] D. Ellis, *Autometrized Boolean Algebras I*, **Canadian Journal of Mathematics**, vol. 3 (1951), pp. 87–93.

[3] D. W. Miller, *On Distance from the Truth as a True Distance*, this **Bulletin**, vol. 6, no. 1 (1977), pp. 15–26.

[4] E. H. Moore, *Introduction to a From of General Analysis*, **The New Haven Mathematical Colloquium** (1910), pp. 1–150.

University of Warwick
England