

Wojciech Dzik, Roman Suszko

ON DISTRIBUTIVITY OF CLOSURE SYSTEMS

A family ζ of subsets of a non-empty set S is said to be a closure system on S if ζ is closed under arbitrary intersections. A closure operator Cn on S is a function from the power set of S into itself such that, for all $X, Y \subseteq S$

$$X \subseteq Cn X = Cn Cn X \text{ and } Cn X \subseteq Cn Y \text{ whenever } X \subseteq Y.$$

A closure operator Cn is said to be finite (or algebraic) if for each $X \subseteq S$, $Cn X = \bigcup \{Cn Y : Y \subseteq X, \text{ finite}\}$.

There is a bijective correspondence between closure operators on S and closure systems on S , namely:

- a) Every closure operator on S defines the closure system

$$\zeta(Cn) = \{X \subseteq S : Cn X = X\},$$

- b) Every closure system ζ on S determines the closure operator

$$Cn_{\zeta} X = \bigcap \{Z \in \zeta : X \subseteq Z\}.$$

A closure system ζ is called inductive if for every upward direct (under set inclusion) $\mathcal{F} \subseteq \zeta$, $\sup \mathcal{F} = \bigcup \mathcal{F} \in \zeta$ ($\mathcal{F}$ is upward directed if for every $X, Y \in \mathcal{F}$ there exists Z such that $X \cup Y \subseteq Z$). A closure operator is finite iff the corresponding closure system is inductive.

Every closure system ζ on S constitutes a complete lattice $\langle \zeta, \subseteq \rangle$ under set inclusion, where

$$\inf \mathcal{F} = \bigcap \mathcal{F}, \sup \mathcal{F} = \inf \{Y \in \zeta : \bigcup \mathcal{F} \subseteq Y\} = Cn_{\zeta}(\bigcup \mathcal{F}),$$

for any $\mathcal{F} \subseteq \zeta$. This lattice need not be distributive. In this note we give necessary and sufficient conditions for some closure systems to be

distributive lattices. (In particular we consider inductive closure systems, i.e. systems given by finite closure operators).

A family $\mathcal{B} \subseteq \zeta$ is said to be a basis of a system ζ if for every $X \in \zeta$ there exists $\mathcal{R} \subseteq \mathcal{B}$ such that $X = \bigcap \mathcal{R}$.

A set $X \in \zeta$ is said to be irreducible (finitely irreducible) in ζ if for every family (finite family) $\mathcal{F} \subseteq \zeta - \{X\}$,

$$X = \bigcap \mathcal{F} \text{ implies } X \in \mathcal{F}.$$

If a system ζ is inductive, then the family of all irreducible sets in ζ constitute the smallest basis of ζ .

The above definitions and results are due to Moore, Birkhoff, Tarski, Ore and Schmidt; see [AL], [LT].

THEOREM 1. *If a closure system ζ has the basis of all (finitely) irreducible sets, then the following conditions are equivalent:*

- (i) $\langle \zeta, \subseteq \rangle$ is a distributive lattice,
- (ii) if $X \cap Y \subseteq Z$, then $X \subseteq Z$ or $Y \subseteq Z$, for every (finitely) irreducible $Z \in \zeta$ and every $X, Y \in \zeta$,
- (iii) $Cn\{a\} \cap Cn\{b\} \subseteq Z$ implies $a \in Z$ or $b \in Z$, for every (finitely) irreducible $Z \in \zeta$ and every $a, b \in S$.

COROLLARY 1. *For every inductive closure system ζ the above conditions (i) - (iii) are equivalent.*

COROLLARY 2. *If Cn is a finite closure operator, then $\langle \zeta(Cn), \subseteq \rangle$ is a distributive lattice iff $Cn\{a\} \cap Cn\{b\} \subseteq Z$ implies $a \in Z$ or $b \in Z$, for any (finitely) irreducible $Z \in \zeta$ and $a, b \in S$.*

For the definitions of Galois representation, dual space and $\mathcal{B}$ -natural dual space we refer to [AL].

A closure system ζ is topological if $\emptyset \in \zeta$ and $X \cup Y \in \zeta$ if $X, Y \in \zeta$ (i.e. ζ is the topological space of closed sets). In this case the corresponding closure operator Cn is additive: $Cn(X \cup Y) = Cn X \cup Cn Y$, $X, Y \subseteq S$, and $Cn \emptyset = \emptyset$.

THEOREM 2. *If ζ is an inductive closure system, then $\langle \zeta, \subseteq \rangle$ is a distributive lattice iff there exists $\mathcal{B}$ -natural dual space for ζ which is topological.*

References

- [LT] G. Birkhoff, *Lattice theory*, **Amer. Math. Soc. Coll. Publ.**, vol. 25 (1967), third edition.
- [AL] D. J. Brown, R. Suszko, *Abstract logics*, **Dissertationes Mathematicae CII** (1973).

*Mathematical Institute
Silesian University
Katowice*

*The Section of Logic
Institute of Philosophy
and Sociology
Polish Academy of Sciences*