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# DEDUCTION THEOREMS FOR *RM* AND ITS EXTENSIONS

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Consider the following Sugihara algebra $\underline{S} = (Q, \rightarrow, \vee, \wedge, \sim)$, where $Q$ is the set of all the rational numbers (including 0) and the operations $\rightarrow, \vee, \wedge, \sim$ are defined as follows:

$$\begin{array}{rcl} \sim x & = & -x \\ x \vee y & = & max(x, y) \\ x \wedge y & = & min(x, y) \\ x \rightarrow y & = & \begin{cases} -x \vee y & \text{if } x \leqslant y, \\ -x \wedge y & \text{otherwise.} \end{cases} \end{array}$$

Let $\underline{S}_a, \underline{S}_2, \underline{S}_3, \underline{S}_4, \ldots$ be subalgebras of $\underline{S}$ generated by the sets $\{0\}$, $\{-1, 1\}$, $\{-1, 0, 1\}$, $\{-2, -1, 1, 2\}, \ldots$ respectively. Put $D = \{x \in Q : x \geqslant 0\}$. Then the pair $(\underline{S}, D)$ is a Sugihara matrix with $D$ as the set of designated elements. We shall also use the matrices $(\underline{S}_1, D_1)$, $(\underline{S}_2, D_2)$,... where $D_i = D \cap |\underline{S}_i|$. The symbol $\underline{S}_i$ will denote both the $i$-element algebra as well as the suitable $i$-element matrix.

Let $\underline{L} = (L, \rightarrow, \vee, \wedge, \sim)$ be a propositional language with $p_1, p_2, \ldots$ as propositional variables. Let $(\underline{A}, B)$ be any matrix under consideration. By $C_{(\underline{A},B)}$ we shall understand the matrix consequence in $\underline{L}$ in the sense of Łoś and Suszko [2], i.e. for every $X \cup \{\alpha\} \subseteq \underline{L}$

$$\alpha \in C_{(\underline{A},B)}(X) \text{ iff } \forall h : \underline{L} \rightarrow^{hom} \underline{A}(hX \subseteq B \Rightarrow h\alpha \in B).$$

To make formulas short we shall write $C_1$ instead of $C_{\underline{S}_i}$. Let us define additional consequences in $\underline{L}$, namely $C_{RM}$ and $C_i^0$ by putting:

$C_{RM}(X) =$ the least set of formulas including $X \cup RM$ and closed under modus ponens and adjunction.

$C_i^0(X) =$ the least set of formulas including $X \cup C_i(\emptyset)$ and closed under modus ponens and adjunction.

The most important results on $RM$ and Sugihara matrices are included in Dunn's [1], it is presupposed here that paper is familiar to the reader.

By Ackermann's rule we mean the one given by the scheme $\alpha, \sim \alpha \vee \beta / \beta$. Moreover, we use new connectives $\supset$ and $\succ$ defined in the following way:

$$\alpha \supset \beta =_{df} [\sim (\alpha \to \sim \beta) \vee (\alpha \to \beta)] \wedge (\sim \alpha \vee \beta),$$
$$\alpha \succ \beta =_{df} \beta \vee (\alpha \to \beta).$$

The following theorems hold:

1. (Dunn [1])
   a. $C_{RM} = C_{\underline{S}}$.
   b. The only systems including $RM$ and closed under substitution, modus ponens and adjunction are those of the form $C_i(\emptyset)$.
   c. $C_1(\emptyset) \supsetneq C_2(\emptyset) \supsetneq C_3(\emptyset) \supsetneq \ldots$
   d. $RM = \bigcap\{C_i(\emptyset) : i \in \omega\}$.
2. $\beta \in C_{2i}(X \cup \{\alpha\})$ iff $\sim \alpha \vee \beta \in C_{2i}(X)$.
3. $\beta \in C_3(X \cup \{\alpha\})$ iff $\sim (\alpha \to \sim \beta) \vee (\alpha \to \beta) \in C_3(X)$.
4. $\beta \in C_{2i+1}(X \cup \{\alpha\})$ iff $\alpha \supset \beta \in C_{2i+1}(X)$.
5. $C_{2i+1} = C^0_{2i+1}$.
6. If $i \geqslant 2$, then $C_{2i} > C^0_{2i}$.
7. Ackermann's rule is not deducible in $C^0_{2i}$, all $i \geqslant 2$.
8. $\beta \in C^0_{2i}(X \cup \{\alpha\})$ iff $\alpha \succ \beta \in C^0_{2i}(X)$.
9. $\beta \in C_{RM}(X \cup \{\alpha\})$ iff $\alpha \supset \beta \in C_{RM}(X)$.

## References

[1] J. M. Dunn, *Algebraic completeness results for R-mingle and its extensions*, **The Journal of Symbolic Logic** 35 (1970), pp. 1–13.

[2] J. Łoś and R. Suszko, *Remarks on sentential logics*, **Indagationes Mathematicae** 20 (1958), pp. 177–183.

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