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A COMPLETENESS PROOF FOR AN INFINITARY TENSE LOGIC

An extended version of this abstract will appear in Theoria.

In his forthcoming examination of G. H. von Wright's tense-logic [2], Krister Segerberg studies certain *infinitary* extensions of the original tense-logic created by von Wright. For one of these extensions, called $W1$, the completeness problem turned out to be harder than was expected at first sight. The purpose of the paper is to give a proof of completeness for $W1$.

We use a countable language of ordinary classical propositional logic together with two modal operators: $\circ$ ("tomorrow") and $\Box$ ("always"). The semantics for tense-logic based on this language uses the frame $\underline{\underline{N}} = (N,', \leqslant)$, where the sucessor-relation is the accessibility relation for $\circ$ and $\leqslant$ for $\Box$, i.e. $\circ(A)$ is true at $n \in N$ iff A is true at the point $n+1$ and $\Box A$ it true at n iff for all $k \geqslant n$, A is true at k. The evaluation of formulae over a model is done according to the familiar Kripke-procedure, with which we assume familiarity. We shall use $\circ^k(A)$ as a short hand for

$$\underbrace{\circ(\circ(\ldots\circ}_{k\ times}(A)\ldots))$$

A set *sum* of formulae from our language *has a model on* $\underline{\underline{N}}$ if there is a model M on the frame $\underline{\underline{N}}$ such that for some point $n \in N$ all the formulae from $\sum$ are true in M.

The main part of [2] is spent on a proof that if a *finite* $\sum$ is consistent in von Wright's system then it has a model on $\underline{\underline{N}}$. Professor Segerberg then notes that $\{-\Box p\} \cup \{\circ^n(p) : n \in N\}$ is consistent in von Wright's logic, because all its finite subsets have models on $\underline{\underline{N}}$ and the rules of the logic are finitary. The whole set, however, has no model on $\underline{\underline{N}}$. To improve on this fact Segerberg then introduces $W1$ as an infinitary natural deduction calculus, cf. [1].

Rules: For every $n \in N$

$$\& I(n) \qquad \frac{\circ^n(A) \quad \circ^n(B)}{\circ^n(A \,\&\, B)} \qquad\qquad \& E(n) \qquad \frac{\circ^n(A_1 \,\&\, A_2)}{\circ^n(A_i)} \quad (i = 1, 2)$$

$$\rightarrow I(n) \qquad \frac{\begin{array}{c}\cancel{\circ^n(A)} \\ \vdots \\ \circ^n(B)\end{array}}{\circ^n(A \rightarrow B)} \qquad\qquad \rightarrow E(n) \qquad \frac{\circ^n(A \rightarrow B) \; \circ^n(A)}{\circ^n(B)}$$

(If desired similar rules for disjunction can be added. In [1] and the paper they are included).

$$-I(n) \qquad \frac{\begin{array}{c}\cancel{A} \\ \vdots \\ \circ^n(B)\end{array} \quad \begin{array}{c}\cancel{A} \\ \vdots \\ \circ^n(-B)\end{array}}{-A} \qquad\qquad -E(n) \qquad \frac{\begin{array}{c}-\cancel{A} \\ \vdots \\ -\circ^n(B)\end{array} \quad \begin{array}{c}-\cancel{A} \\ \vdots \\ -\circ^n(-B)\end{array}}{A}$$

$$\Box I(n) \qquad \frac{(\circ^{n+k}(A))_{k \in N}}{\circ^n(\Box A)} \qquad\qquad \Box E(n) \qquad \frac{\circ^n(\Box A)}{\circ^{n+k}(A)} \quad \text{all } k \in N$$

We prove the following Theorem, first stated without proof by Segerberg: If $\sum$ is consistent in $W1$, then it has a model on $\underline{\underline{N}}$. The proof is very similar to a Henkin-type proof for $L_{\omega_1\omega}$. One uses the fundamental

LEMMA. *If $\sum$ is consistent in $W1$, then, for some $k \in N$, so is $\sum \cup \{\circ^{n+k}(A) \rightarrow \circ^n(\Box A)\}$.*

With the aid of the lemma a consistent superset of $\sum$ is then constructed in such a fashion it has all the properties needed for a "canonical model" – proof.

Finally we note that by dropping the negation from our language and instead adding absurdity $\perp$ as a primitive with the rules

$$\circ^n(\perp) \qquad \cfrac{\begin{array}{c}\not{A \to \perp} \\ \vdots \\ \circ^n(\perp)\end{array}}{A}$$

one will get a system easily seen to be mutually interpretable with $W1$ and for which one can prove a normalization theorem along the lines of Prawitz [1]. The remarkable ease with which the natural deduction methods work should be credited to the great similarity between $W1$ and elementary number theory with the omega-rule, as it is well known that a smooth proof theory exists for the latter.

References

[1] Dag Prawitz, **Natural deduction: A proof-theoretical study**, Stockholm: Almqvist & Wiksell, 1965.

[2] Krister Segerberg, *"von Wright's tense logic"*, forthcoming in **The philosophy of G. H. von Wright**, to be edited by P. A. Schilpp.

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