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DEGREES OF MAXIMALITY OF ŁUKASIEWICZ-LIKE SENTENTIAL CALCULI

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Łukasiewicz-like n-valued sentential calculi

The algebra of formulas

$$\underline{L} = (L, \to, \vee, \wedge, \neg) \tag{1}$$

will be called the *language of Łukasiewicz-like sentential calculi* ($\to, \vee, \wedge, \neg$ denote the implication, disjunction, conjunction and negation connectives, respectively). Now consider the algebras

$$\underline{A}_n = (A_n, \to, \vee, \wedge, \neg) \tag{2}$$

where n is an arbitrary natural number, $n \geqslant 2$, similar to $\underline{L}$ and defined as follows:

(i) $A_n = \{0, {}^1/_{n-1}, \ldots, {}^{n-2}/_{n-1}, 1\}$
(ii) $x \to y = min(1, 1 - x + y)$; $x \vee y = max(x, y)$
$x \wedge y = min(x, y)$; $\neg x = 1 - x$
for $x, y \in A_n$

Given $n \geqslant 2$, let $I \subsetneq A_n$ be any subset of A_n such that $1 \in I$ and $0 \notin I$. Then the matrix

$$M_n^I = (\underline{A}_n, I) \tag{3}$$

will be called the *n-valued Łukasiewicz-like matrix of the type I*. The symbol C_n^I will be used to denote the consequence operation determined by the matrix M_n^I (in $\underline{L}$).

The pair

$$Ł_n^I = (\underline{L}, C_n^I) \tag{4}$$

will be called the *n-valued Łukasiewicz-like sentential calculus of the type I*. Clearly $M_n^{\{1\}} = M_n$, where M_n is the well known n-valued Łukasiewicz matrix and $Ł_n^{\{1\}} = Ł_n$ appears to be the n-valued Łukasiewicz calculus.

Given $n \geqslant 2$, the class of all n-valued Łukasiewicz-like sentential calculi is a superclass of Rosser Turquette's [2] class of generalizations of the n-valued Łukasiewicz calculus. Recall that for every matrix M_n^I in [2] $I = \{x \in A_n : x \geqslant {}^{i}\!/_{n-1}\}$, where ${}^{i}\!/_{n-1}$ is some given element of A_n not equal to 0.

2. $Ł_n^I$-algebras

For the whole present section let us assume that $Ł_n^I = (\underline{L}, C_n^I)$ is some fixed n-valued Łukasiewicz-like sentential calculus. For every $x, y \in A_n$ let us put

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leqslant y \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Clearly, $\Rightarrow$ is definable in M_n^I, cf. [2]. The same symbol $\Rightarrow$ will be used to denote an appropriate sentential connective in $\underline{L}$.

Now, let us consider matrices $M = (\underline{A}_M, I_M)$, where $\underline{A}_M = (A_M, \rightarrow, \vee, \wedge, \neg)$ are algebras similar to $\underline{L}$ and $I_M \subseteq A_M$. Given such M, by Cn_M we shall denote the consequence operation determined by M (in $\underline{L}$). The symbol $Matr(C_n^I)$ will be used to denote the class of all C_n^I-matrices, i.e. the class of matrices M, such that $C_n^I \leqslant Cn_M$.
For every $M \in Matr(C_n^I)$ we put

$$a \approx_M b \text{ if and only if } a \Rightarrow b, b \Rightarrow a \in I_M. \tag{6}$$

and

$$V_M = \{a \in A_M : \text{ there exists a formula } \alpha \in L \text{ such that } j_1(\alpha) \in C_n^I(\emptyset) \text{ and such that there exists a homomorphism } h : \underline{L} \to \underline{A}_M \text{ for which } h\alpha = a\}, \tag{7}$$

where j_1 is a sentential connective corresponding to Rosser and Turquette's j_1 ($j_1(x) = 1$ whenever $x = 1$ and $j_1(x) = 0$ otherwise), cf. [2].

Lemma 1.

(i) The relation $\approx_M$ is a congruence of M

(ii) $V_{M/\approx_M}$ is a one element set

(iii) Put $1_M = V_{M/\approx_M}$ (see (ii)). Then if $|a| \to |b| = 1_M$ and $|b| \to |a| = 1_M$, then $|a| = |b|$.

In the sequel we shall consider the class

$$Matr^R(C_n^I) = \{{}^M/_{\approx_M} : M \in Matr(C_n^I)\}. \tag{8}$$

3. The equational characterization of $Matr^R(C_n^I)$

Given finite $n \geqslant 2$, let $HSP(\underline{A}_n)$ denote the smallest equational class of algebras containing the algebra $\underline{A}_n$. Now let us denote by $HSP(M_n^I)$ a corresponding class of matrices defined in the following manner: For every $\underline{A}_M \in HSP(\underline{A}_n)$ we take the matrix $M = (\underline{A}_M, I_M)$, where I_M is determined by one of the following conditions:

(i_1) If $\underline{A}_M = \underline{A}_M$, then $I_M = I$

(i_2) If $\underline{A}_M$ is a product of the indexed set of algebras $\{\underline{A}_{M_i} : i \in T\}$, then $I_M = \sqcap\{I_{M_i} : i \in T\}$

(i_3) If $\underline{A}_M$ is a subalgebra of some algebra $\underline{A}_{M_0}$, then $I_M = \underline{A}_M \cap I_{M_0}$

(i_4) If $\underline{A}_M = h(\underline{A}_{M_0})$, where h is a homomorphism, then $I_M = h(I_{M_0})$.

In the sequel, the element of $\underline{A}_M \in HSP(\underline{A}_n)$ being a natural counterpart of $1 \in A_n$ will be denoted by 1_M (cf. Lemma 1).

Let us now assume that $I = \{m_1, \ldots, m_k\}$, and consider Rosser and Turquette's functions $j_{m_1}, \ldots, j_{m_k}$, i.e. such functions of $\underline{A}_n$, that

$$j_{m_i}(x) = \begin{cases} 1 & \text{if } x = m_i \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

cf. [2]. Because j_{m_i}'s are definable in $\underline{A}_n$, we can assume $j_{m_i}(\alpha)$, where $\alpha \in L$, to be formulas of $\underline{L}$.

In turn, let us put

$$\varphi_I(\alpha) = j_{m_1}(\alpha) \vee \ldots \vee j_{m_k}(\alpha); \tag{10}$$

obviously $\varphi_I(\alpha) \in L$. Moreover, for any $x \in A_n$ we have

$$\varphi_I(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

From the representation theorem for $HSP(\underline{A}_n)$ and the properties of φ_I, we obtain

LEMMA 2. *$M = (\underline{A}_n, I_M) \in HSP(M_n^I)$ if and only if 1^0. $\underline{A}_M \in HSP(\underline{A}_n)$ and 2^0. $I_M = \{a \in A_M; \varphi_I(a) = 1_M\}$.*

In turn, using the results of [3] and Lemma 2, one can prove

LEMMA 3. *$HSP(M_n^I) \subseteq Matr^R(C_n^I)$.*

Now, we are going to construct Lindenbaum matrix for $Ł_n^I$. In the language $\underline{L}$ we introduce the following relation:

$$\alpha \approx \beta \text{ if and only if } \alpha \Rightarrow \beta, \beta \Rightarrow \alpha \in C_n^I(\emptyset). \tag{12}$$

One can easily verify that $\approx$ is a congruence relation on L. The quotient matrix

$$\Lambda_n = (\underline{L}/_\approx, Cn_n^I(\emptyset)/_\approx) \tag{13}$$

will be called the *Lindenbaum matrix for* $Ł_n^I$.

LEMMA 4. *$\underline{L}/_\approx$ is free in the class $\{\underline{A}_M : (\underline{A}_M, I_M) \in Matr(C_n^I)$ for some $I_M \subseteq A_M\}$, the set $\{||p|| : p \in V\}$ (where V is the set of all sentential* *variables in* $\underline{L}$*) being the set of free generators in* $\underline{L}/_{\approx}$*. If* $p_1 \neq p_2$*, then* $||p_1|| \neq ||p_2||$.

Using the last lemma and Lemma 2, one can prove the following

Lemma 5. $Matr^R(C_n^I) \subseteq HSP(M_n^I)$.

Combining the results of Lemma 3 and Lemma 5 we obtain the main theorem of the present section.

Theorem 1. $Matr^R(C_n^I) = HSP(M_n^I)$.

4. Degrees of maximality of $Ł_n^I$

Given a sentential calculus $(\underline{L}, C)$, $\underline{L}$ being a sentential language C a structural consequence operation on $\underline{L}$, the *degree of maximality* of $(\underline{L}, C)$, $dm(\underline{L}, C)$, is equal to the cardinal number of all structural consequence operations C' such that $C \leqslant C'$ (cf. e.g. [4]). Let us now consider any Łukasiewicz-like calculus $Ł_n^I$. By the argument in [4] we obtain

$$dm(Ł_n^I) = card\{Cn_M : M \subseteq Matr(C_n^I)\}. \tag{14}$$

Let $M \in Matr(C_n^I)$ be an arbitrary matrix and let $\approx_M$ be the congruence relation defined by (6). From the fact that $\approx_M$ is a congruence of M (Lemma 1 (i)), it follows that

$$Cn_M = Cn/_{\approx_M} \tag{15}$$

As it was defined in Section 2, the whole class of such $M/_{\approx_M}$ is $Matr^R(C_n^I)$. Moreover, according to the equational characterization given in Section 3 we have $Matr^R(C_n^I) = HSP(M_n^I)$. This implies that the sequence operation determined by the matrix belonging to the class $HSP(M_n^I)$. Consequently,

$$dm(Ł_n^I) = card\{Cn_M : M \subseteq HSP(M_n^I)\}. \tag{16}$$

Using Lemma 9 in [4] and a correspondence between $HSP(M_n^I)$ and $HSP(M_n)$ (cf. Lemma 2), one can prove the following:

LEMMA 6. *For each $M \in HSP(M_n^I)$ there are pairwise different submatrices $N_1, \ldots, N_s$ of M_n^I such that $Cn_M = Cn_{N_1 \times \ldots N_s}$.*

In turn, the following theorem is an easy corollary to (16) and Lemma 6.

THEOREM 2. *The degree of maximality of any (finite) Łukasiewicz-like sentential calculus $Ł_n^I$ is finite.*

REMARK. A more close inspection of the classes $HSP(M_n^I)$ and $HSP(M_n)$ leads to the conclusion that for any given $n \geqslant 2$, the degrees of maximality of the calculi $Ł_n$ and $Ł_n^I$ are equal. In particular for the calculi $Ł_n$, $Ł_n^I$ for which $(n-1)$ is prime, the degree of maximality equals 4 (cf. [1]).

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