Bulletin of the **S**ection of **L**ogic
Volume 6/2 (1977), pp. 92–95

reedition 2011 [original edition, pp. 92–92]

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REMARKS ON M^n-COUNTERPARTS OF SOME NORMAL CALCULI

This is an abstract of the paper submitted to Studia Logica.

By M^n-counterpart of any modal system we mean the set of all formulas which, while preceded n-times by sign M, become theses of the system. In this paper, for some normal modal systems we construct the greatest normal modal systems with equal M^n-counterparts.

We use well – known logical and set – theoretical notation. The symbol ω denotes the set of natural numbers; the elements of this set will be denoted by n, m, k. The logical connectives will be represented by $\rightarrow, L, M$ denoting material implication, necessity and possibility, respectively. The formulas will be represented by capitals $A, B, \ldots$. By for we denote the set of all formulas. We put

$$L^0A = A,\ L^{n+1}A = LL^nA,\ M^0A = A,\ M^{n+1}A = MM^nA.$$

Let PC denote the set of all classical tautologies. Cn_R is a consequence operator defined by PC and a set R of rules of deduction, whereas Cn_{R_0} is defined by means of PC, detachment and Gödel's rule: if A, then LA. By sb we mean the rule of substitution.

Let

$K = Cn_{R_0}(L(A \rightarrow B) \rightarrow (LA \rightarrow LB))$,
$D = Cn_{R_0}(K, M(A \rightarrow A))$.

As is well known, the system K (see [2]) is the smallest normal modal system and D is a denotic system of Lemmon (see [3]). By easy computation we have

LEMMA 1. *The following formulas are thesis of the system D.*

(1) $L^nA \to M^nA$,
(2) $M^n(A \to A) \to M(A \to A)$,
(3) $M(A \to A) \to M^n(A \to A)$.

We put

$$\begin{aligned} \mathcal{N} = &\ \{X : SbCn_{R_0}(X) = X\}, \\ \mathcal{N}_X = &\ \{Y : SbCn_{R_0}(Y) = Y\}, \quad \text{where } X \in \mathcal{N}, \\ M^n - X = &\ \{A \in FOR : M^nA \in X\}; \quad \text{where } X \in \mathcal{N},\ n \in \omega. \end{aligned}$$

THEOREM 2. *For every $S \in \mathcal{N}$ and $n \in \omega$ the following conditions are equivalent:*
(1) $S \subset M^n - S$,
(2) $D \subset S$,
(3) $M^n - S \neq \emptyset$.

The proof is analogous to that of Theorem 4 in [4].

COROLLARY 3. *D is the weakest normal modal system for which $M^n - D \neq \emptyset$, where $n = 1, 2, \ldots$.*

Now, for certain normal modal systems we are going to define the greatest ones, whose M^n-counterpart coincide.

We shall use the following deduction rules:

(r_1^{nk}): If M^nL^kA, then $M^nL^{k+1}A$,
(r_2^{nk}): If $M^nL^kM^nA$, then M^nA,
(r_3^{nk}): If M^nL^kA, $M^nL^k(A - B)$, then M^nL^kB.

DEFINITION 4. $\zeta_n^k = \{S \in \mathcal{N}_D : (r_1^{nk}), (r_2^{nk}), (r_3^{nk})$ are permissible in $S\}$.

DEFINITION 5. $\zeta_n = \bigcup_{k \geqslant 1} \zeta_n^k$.

Notice that if $S \in \zeta_n$, then there exists a natural number k such that $S \in \zeta_n^k$. Let $k(S)$ denote one of those natural numbers for which $S \in \zeta_n^{k(S)}$.

Let us restrict our consideration to the family of normal logics S such that $S \in \zeta_n$.

THEOREM 6. *Let $S \in \zeta_n$ and $S_1 = \{A \in FOR : M^nL^{k(S)}A \in S\}$. S_1 is the greatest normal modal system for which $M^n - S_1 = M^n - S$.*

From this theorem we obtain

COROLLARY 7. *Let $S \in \zeta_n$ and let Z be a normal modal system for which $M^n - S = M^n - Z$. Then $Z \in \zeta_n$.*

COROLLARY 8. *Let $S, Z \in \zeta_n$. The following conditions are equivalent:*

(1) Z is the greatest normal modal system for which $M^n - Z = M^n - S$,

(2) Rule: If $M^n L^{k(S)} A$, then A is permissible in Z.

Let S be any normal modal system. It is known that for S there exists a set $\mathcal{A}_S$ of axioms (finite or infinite) and a finite set R_S of rules of deduction such that $S = Cn_{R_S}(Sb\mathcal{A}_S)$. Notice that R_S contains merely the detachment rule for material implication and the Gödel's rule. Thus we can assume that $S = Cn_{R_0}(Sb\mathcal{A}_S)$.

THEOREM 9. *Let $S = Cn_{R_0}(Sb\mathcal{A}_S)$ be a normal modal system from ζ_n and let S_1 be the greatest normal modal system for which $M^n - S_1 = M^n - S$. Then $S_1 = Cn_{R_0 \cup \{\frac{M^n L^{k(S)} A}{A}\}}(Sb\mathcal{A}_S)$.*

Let $S = Cn_{R_0}(Sb\mathcal{A}_S)$ be any normal modal system belonging to ζ_n. Notice that by virtue of Theorem 9 the problem of axiomatization of $M^n - S$ is equivalent to the problem of axiomatizing $M^n - S_1$, where S_1 is the greatest normal modal system for which $M^n - S_1 = M^n - S$.

We shall use the following deduction rules:

(R_1^k): If $L^k A$, then $L^{k+1} A$,
(R_2^k): If $L^k A$, $L^k(A \to B)$, then $L^k B$,
(R_3^{kn}): If $L^k M^n L^k A$, then $L^k A$,
(R_4^{kn}): If $L^k M^n A$, then A.

THEOREM 10. *For every $S \in \zeta_n$ $M^n - S = Cn_{R^n}(SbL^{k(S)}\mathcal{A}_S)$, where $L^k \mathcal{A}_S = \{L^k A : A \in \mathcal{A}_S$, $R^n = \{R_1^{k(S)}, R_3^{k(S)n}, R_3^{k(S)n}, R_4^{k(S)n}\}$.*

References

[1] J. J. Błaszczuk, W. Dziobiak, *An Axiomatization of M^n-counterparts for some modal logics*, **Reports on Mathematical Logic** 6 (1976), pp. 3–6.

[2] M. J. Creswell, G. E. Hughes, **An Introduction to Modal Logic**, London 1968.

[3] E. J. Lemmon, *Algebraic semantics for modal logic I*, **The Journal of Symbolic Logic** 31 (1966), pp. 46–65.

[4] J. Perzanowski, *On M-fragments and L-fragments of normal modal propositional logics*, **Reports on Mathematical Logic** 5 (1975), pp. 63–72.

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