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ON THE DEPTH OF A CONSEQUENCE OPERATION

In this paper we define a concept of depth of a consequence operation which seems to have a few useful properties. To make our definition worthwhile we shall show that the concept of depth leads to a strengthening of the well-known theorem of R. Wójcicki [4]. For unexplained terminology and notations we refer the reader to R. Wójcicki [5]. Algebras and matrices considered in this paper are of the same similarity type τ indicating a sequence (possibly infinite) of finitary operations. Boldface Latin capitals will be used for ramified matrices, the corresponding German capitals for algebras of these matrices and the corresponding Latin capitals for carriers of these algebras, for example $\mathbb{M}$ denotes a ramified matrix $\langle \mathfrak{M}, D\rangle$ where $\mathfrak{M}$ is an algebra $\langle M, \Omega\rangle$ and D is a family of subsets of M. The notation $\mathfrak{F} = \langle F, \Omega\rangle$ will be reserved for absolutely free algebra of terms free-generated by an infinite set of variables $Var = \{z_1, z_2, \ldots\}$ and the notation $\mathfrak{F}_n = \langle F_n, \Omega\rangle$ for the subalgebra of $\mathfrak{F}$ generated by $\{z_1, \ldots, z_n\}$, $n = 1, 2, \ldots$. If $\alpha \in F$ then $Var(\alpha)$ denotes the set of variables occurring in α, i.e. the smallest subset of Var that generates a subalgebra of $\mathfrak{F}$ containing α. Recall that a retraction of $\mathfrak{F}$ is an endomorphism $p \in Hom(\mathfrak{F}, \mathfrak{F})$ such that $p \circ p = p$ (see [2]). By a reduction of a set of variables $X \subseteq Var$ we mean a retraction p of $\mathfrak{F}$ such that $pX \subseteq X$. We say that p is n-reduction of X $(n = 1, 2, \ldots)$ if $|pX| \leqslant n$. The symbol $Rd_n(X)$ denotes the set of all n-reductions of X. We say that n-reductions are conclusive for a consequence operation C on F iff $\alpha \in C(\{p(\alpha) : p \in Rd_n(Var(\alpha))\})$ for every $\alpha \in F$. Let ∇_n be the set of all consequence operations on F for which n-reductions are conclusive and let $\mathbb{V}_n$ be the set of all structural consequence operations of ∇_n. The reader will find no difficulty in verifying that $\nabla_n \subseteq \nabla_{n+1}$, $\mathbb{V}_n \subseteq \mathbb{V}_{n+1}$ for every $n = 1, 2, \ldots$; each ∇_n is a principal filter of the lattice of all consequence operations on F and each $\mathbb{V}_n$ is a principal filter of the lattice of all structural consequence operations on F. Let us denote by r_n the rule composed of all sequents of the form $\langle\{p(\alpha) : p \in Rd_n(Var(\alpha))\}, \alpha\rangle$ where $\alpha \in F$ and by $\mathbf{r}_n$ the smallest structural rule containing r_n i.e. the rule of all sequents $\langle\{\varepsilon(p(\alpha)) : p \in Rd_n(Var(\alpha)), \varepsilon(\alpha))\rangle$ where $\alpha \in F$ and $\varepsilon \in Hom(\mathfrak{F}, \mathfrak{F})$. Let Γ_n, $\mathbf{\Gamma}_n$ be consequence operations on F determined by the rules r_n, $\mathbf{r}_n$ respectively. Then we get the following:

PROPOSITION 1.

(i) Γ_n, $\mathbf{\Gamma}_n$ *are the smallest elements of* ∇_n, $\mathbb{V}_n$ *respectively;*

(ii) $\Gamma_n(\alpha) = \mathbf{\Gamma}_n(\alpha) = \{\alpha\}$ *for every* $\alpha \in F$ *and* $n = 1, 2, \ldots$*;*

(iii) If every operation in Ω *is either nullary or unary then* $\Gamma_n(X) = \mathbf{\Gamma}_n(X) = X$ *for every* $X \subseteq F$ *and* $n = 1, 2\ldots$*;*

(iv) If some operation in Ω *is neither nullary nor unary then* $\Gamma_n > \Gamma_{n+1}$, $\mathbf{\Gamma}_n > \mathbf{\Gamma}_{n+1}$ *for every* $n = 1, 2, \ldots$ *and* $\bigwedge_{1 \leqslant n < \omega} \Gamma_n(X) = \bigwedge_{1 \leqslant n < \omega} \mathbf{\Gamma}_n(X) = X$ *for every* $X \subseteq F$.

Given a consequence operation C on F, we define the depth of C as the smallest ordinal of the set $\{n : C \in \nabla_n\} \cup \{\omega\}$. The depth of C will be denoted by $dp(C)$. Let us note the following:

PROPOSITION 2.

(i) If $C \leqslant C'$*, then* $dp(C) \geqslant dp(C')$*;*

(ii) If every operation in Ω *is either nullary or unary then* $dp(\Gamma_n) = dp(\mathbf{\Gamma}_n) = 1$ *for every* $n = 1, 2, \ldots$*;*

(iii) If some operation in Ω *is neither nullary nor unary then* $dp(\Gamma_n) = dp(\mathbf{\Gamma}_n) = n$ *for every* $n = 1, 2, \ldots$ *and* $dp(\bigwedge_{1 \leqslant n < \omega} \Gamma_n) = dp(\bigwedge_{1 \leqslant n < \omega} \mathbf{\Gamma}_n) = \omega$.

Given a ramified matrix $\mathbb{M} = \langle \mathfrak{M}, D\rangle$, it is known (see [3]) that $\mathbb{M}$ determines a structural consequence operation $C_{\mathbb{M}}$ on F if one puts: $\alpha \in C_{\mathbb{M}}(X)$ iff $vX \subseteq I$ implies $v(\alpha) \in I$ for every $I \in D$ and $v \in Hom(\mathfrak{F}, \mathfrak{M})$. An useful property of the concept of depth is expressed by the following:

PROPOSITION 3. $dp(C_{\mathbb{M}}) \leqslant |M|$.

Recall that a congruence of a ramified matrix $\mathbb{M} = \langle \mathfrak{M}, D\rangle$ ($\mathbb{M}$-congruence) is any congruence Θ of the algebra $\mathfrak{M}$ satisfying: $[a]_\Theta \subseteq \bigcap(I \in D : a \in I)$

for every $a \in M$. It is easy to see that $\mathbb{M}$-congruences form a principal ideal of the lattice of all congruences of $\mathfrak{M}$. Let us note the following simple observations:

Proposition 4. *Let $\mathbb{M} = \langle \mathfrak{M}, D \rangle$ be a ramified matrix, let D' be the smallest closure system on M containing D and let $D'' = \{\bigcap(I \in D : a \in I) : a \in M\}$. Put $\mathbb{M}' = \langle \mathfrak{M}, D' \rangle$ and $\mathbb{M}'' = \langle \mathfrak{M}, D'' \rangle$, then the following conditions hold:*

(i) $\mathbb{M}, \mathbb{M}'$ *and* $\mathbb{M}''$ *-congruences are the same,*
(ii) $C_{\mathbb{M}} = C_{\mathbb{M}'} \leqslant C_{\mathbb{M}''}$,
(iii) $C_{\mathbb{M}/\Theta} = C_{\mathbb{M}}$ *whenever* Θ *is* M*-congruence (see [5]).*

Let us fix the notation $\Theta_{\mathfrak{M}}$ for the invariant congruence of $\mathfrak{F}$ determined by an algebra $\mathfrak{M}$ i.e. $\Theta_{\mathfrak{M}} = \bigcap(ker(v) : v \in Hom(\mathfrak{F}, \mathfrak{M}))$ $\Theta^n_{\mathfrak{M}} = \Theta_{\mathfrak{M}} \cap (F_n \times F_n)$, $n = 1, 2, \ldots$. For every consequence operation C on F we define a congruence of $\mathfrak{F}$ associated with C as the greatest congruence of the ramified matrix $\langle \mathfrak{F}, \{C(X) : X \subseteq F \rangle$. The congruence of $\mathfrak{F}$ associated with C will be denoted by Θ_C, $\Theta^n_C = \Theta_C \cap (F_n \times F_n)$, $n = 1, 2, \ldots$. It is easy to see that $\Theta_{\mathfrak{M}} \leqslant \Theta_{C_{\mathbb{M}}}$ for every ramified matrix $\mathbb{M}$ and if $C \leqslant C'$ then $\Theta_C \leqslant \Theta_{C'}$. Now we shall state a strengthening of the theorem of R. Wójcicki [4] promised at the beginning:

Proposition 5. *Let C be a structural consequence operation on F such that the algebra $\mathfrak{F}_n/\Theta^n_C$ is finite and $dp(C) \leqslant n$ for some $n = 1, 2, \ldots$. Then the number of invariant theories of C is finite and every such a theory has a finite matrix adequate.*

We say that a consequence operation C is hereditarily infinite if there is no finite ramified matrix $\mathbb{M}$ such that $C_{\mathbb{M}} \leqslant C$. The results stated above yields the following:

Criterion. *Every consequence operation C on F satisfying (a) or (b) must hereditarily infinite:*

(a) $\mathfrak{F}_n/\Theta^n_C$ *is infinite for some* $n = 1, 2, \ldots$,
(b) $dp(C) = \omega$.

Let us note that the conditions (a), (b) are independent. Each the consequence operation $\mathbf{\Gamma}_n$ satisfies (a) but not (b) and the structural consequence operation of the intermediate logic LC of M. Dummett [1] satisfies

(b) but not (a). The structural consequence operation of the intuitionistic propositional logic satisfies both (a) and (b). On the other hand it is possible to construct a matrix $\mathbb{N}$ of type $\langle 2 \rangle$ with $C_{\mathbb{N}}$ satisfying neither (a) nor (b) and such that $C_{\mathbb{N}} \neq C_{\mathbb{M}}$ for every finite ramified matrix $\mathbb{M}$.

PROBLEM. Characterize hereditarily infinite consequence operations.

References

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