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AN AXIOMATIZATION OF THE VARIETY OF EQUIVALENTIAL ALGEBRAS BY A SINGLE IDENTITY

This is a summary of the lecture read at the seminar held by Andrzej Wroński in Cracow, March 1977.

The notion of equivalential algebra introduced in [1] is an algebraic counterpart of the equivalential fragment of the intuitionistic propositional logic. This fragment was axiomatized by R. E. Tax [2] by means of the single axiom $((q \leftrightarrow (q \leftrightarrow p)) \leftrightarrow ((q \leftrightarrow (q \leftrightarrow p)) \leftrightarrow (p \leftrightarrow (p \leftrightarrow (r \leftrightarrow s))))) \leftrightarrow ((p \leftrightarrow s) \leftrightarrow (r \leftrightarrow p))$ and the following two rules of inference $p, p \leftrightarrow q \vdash q$, $q \vdash p \leftrightarrow (p \leftrightarrow q)$. The class of equivalential algebras was defined in [1] as the variety of all algebras of type $\langle 2 \rangle$ satisfying the identities

($e1$) $(a \leftrightarrow a) \leftrightarrow b = b$,
($e2$) $((a \leftrightarrow b) \leftrightarrow c) \leftrightarrow c = (a \leftrightarrow c) \leftrightarrow (b \leftrightarrow c)$,
($e3$) $((a \leftrightarrow b) \leftrightarrow ((a \leftrightarrow c) \leftrightarrow c)) \leftrightarrow ((a \leftrightarrow c) \leftrightarrow c) = a \leftrightarrow b$.

From now we adopt the convention of associating to the left and ignoring the equivalence sign $\leftrightarrow$. For example the identities ($e1$), ($e2$), ($e3$) should be abbreviated as follows:

($e1$) $aab = b$,
($e2$) $abcc = ac(bc)$,
($e3$) $ab(acc)(acc) = ab$.

In this paper it is proved that the variety of equivalential algebras can be defined by means of the single identity:

(e) $yz(xz)(xyzz)(aaa(abb)(abb(ddbb))(c(abb)(abb)c))(uu) = a$.

If $p, p_1, \ldots, p_n$ are terms of type $\langle 2\rangle$ and $x_1, \ldots, x_n$ are pairwise distinct variables, then by $p[x_1/p_1, \ldots, x_n/p_n]$ we denote the result of simultaneous substitution of p_i for x_i $(i = 1, \ldots, n)$ in p.

The reader will find no difficulty in verifying that the identity (e) defines the variety of equivalential algebras with the help of the following

Sketch of Proof.

Introduce the following abbreviations:

$$a^1 = aa(aa)(aa)(aa(aa)(aa))(aa(aa)(aa)(aa(aa)(aa)))-$$
$$-(aa(aa)(aa)(aa(aa)(aa))(aa(aa)(aa))(aa(aa)(aa))),$$

$$a^{i+1} = a^i[a/a^1] \quad \text{for every} \quad i = 1, 2, \ldots.$$

Observing that

$$yz(xz)(xyzz)[x/aa(aa)(aa), y/aa(aa)(aa), zaa(aa)(aa)] = a^1 =$$
$$= aaa(abb)(abb(ddbb))(c(abb)(abb)c)[a/aa, b/aa, c/aa(aa)(aa), d/a]$$

one can infer from (e) the following identities:

(1) $a^1a^1(uu) = aa$,
(2) $a^{i+1}a^{i+1}(uu) = a^ia^i$ for every $i = 1, 2, \ldots$.

Now let us compute: $aa =_{(1)} a^1a^1(uu) =_{(2)} a^2a^2(aa)(uu) =_{(1)}$ $a^2a^2(a^1a^1(aa))(uu) =_{(2)} a^2a^2(a^2a^2(a^2a^2)(aa))(uu) =_{(1),(2)}$ $a^5a^5(aa)(a^3a^3(aa))(a^3a^3(a^5a^5)(aa)(aa))(a^5a^5(a^5a^5)(a^5a^5)(a^5a^5(aa)-$ $-(aa))(a^5a^5(aa)(aa)(a^2a^2(aa)(aa)))(a^3a^3(a^5a^5(aa)(aa))(a^5a^5(aa)-$ $-(aa))(a^3a^3)))(uu) =_{(e)} a^5a^5 =_{(e)} b^5b^5(aa)(a^3a^3(aa))(a^3a^3(b^5b^5)-$ $-(aa)(aa))(a^5a^5(a^5a^5)(a^5a^5)(a^5a^5(aa)(aa))(a^5a^5(aa)(aa)(a^2a^2(aa)-$ $-(aa)))(a^3a^3(a^5a^5(aa)(aa))(a^5a^5(aa)(aa))(a^3a^3)))(uu) =_{(1),(2)}$ $b^2b^2(a^2a^2(a^2a^2)(aa))(uu) =_{(2)} b^2b^2(a^1a^1(aa))(uu) =_{(1)}$ $b^2b^2(aa)(uu) =_{(2)} b^1b^1(uu) =_{(1)} bb$ thus we have proved

(3) $aa = bb$.

Put $\mathbf{1} = aa$ (the definition is correct by virtue of (3)). We infer successively the following conditions:

(4)	$\mathbf{1}\nVdash = \mathbf{1}$	by the definition of **1**,
(5)	$yz(xz)(xyzz)\mathbf{11} = \mathbf{1}$	by (e), (4),
(6)	$\mathbf{1}(\mathbf{1}xx)\mathbf{11} = \mathbf{1}$	by (5),
(7)	$\mathbf{1}(x\mathbf{1})(x\mathbf{111})\mathbf{11} = \mathbf{1}$	by (5),
(8)	$\mathbf{1}(\mathbf{1}a(abb)(abb(\mathbf{1}bb))(c(abb)(abb)c))\mathbf{1} = a$	by (e),
(9)	if $a\mathbf{11} = \mathbf{1}$ then $a\mathbf{1} = \mathbf{1}$	by (6), (7),
(10)	if $a\mathbf{1} = \mathbf{1}$ then $a = \mathbf{1}$	by (8), (9),
(11)	if $a\mathbf{11} = \mathbf{1}$ then $a = \mathbf{1}$	by (9), (10),
(12)	$\mathbf{1}(\mathbf{1}xx) = \mathbf{1}$	by (6), (11),
(13)	$yz(xz)(xyzz) = \mathbf{1}$	by (5), (11),
(14)	$\mathbf{1}xx = \mathbf{1}$	by (8), (12),
(15)	$\mathbf{1}(\mathbf{1}a(abb)(abb\mathbf{1})(c(abb)(abb)c))\mathbf{1} = a$	by (8), (14),
(16)	$\mathbf{1}(\mathbf{1}a(abb)(abb\mathbf{1})\mathbf{1})\mathbf{1} = a$	by (14), (15),
(17)	$\mathbf{1}(\mathbf{1}(\mathbf{1}a\mathbf{1})\mathbf{1})\mathbf{1} = \mathbf{1}a$	by (14), (16),
(18)	$\mathbf{1}(\mathbf{1}(\mathbf{1}a\mathbf{1})\mathbf{1})\mathbf{1} = a$	by (16),
(19)	$\mathbf{1}a = a$	by (17), (18),
$(e1)$	$aab = b$	by (19),
(20)	$a\mathbf{111} = a$	by (18), (19),
(21)	$a\mathbf{11}a = \mathbf{1}$	by (10), (14), (15),
(22)	if $ab = \mathbf{1}$ then $a = b$	by (19), (21),
(23)	$a\mathbf{11} = a\mathbf{1}$	by (21), (22),
(24)	$a\mathbf{1} = a$	by (20), (23),
(25)	$yz(xz) = xyzz$	by (13), (22),
(26)	$ab = ba$	by (23), (24), (25),
$(e2)$	$xz(yz) = xyzz$	by (25), (26),
(27)	$abbb = ab$	by (19), (25), (26),
(28)	$a(abb)(abb) = a$	by (16), (19), (24),
(29)	$xzz(yzz) = xyzz$	by $(e2)$, (27),
(30)	$abb(baa) = ab$	by (19), (26),
(31)	$abbabb = \mathbf{1}$	by $(e2)$, (27),
(32)	$abbab = b$	by (22), (31),
(33)	$ab(baa) = abb$	by $(e1)$, $(e2)$, (26)-(29), (32),
(34)	$caabb = cbbaa$	by (26), (27), (29), (30), (33),
(35)	$caa(abb)(abb)(caa) = \mathbf{1}$	by (15), (24), (26), (28), (29), (34),
(36)	$caa(abb)(abb) = caa$	by (22), (35),
$(e3)$	$ac(abb)(abb) = ac$	by (26), (27), (36).

Thus we have proved that the identities $(e1)$, $(e2)$, $(e3)$ can be derived from (e). The proof of the converse is easy and therefore is omitted.

References

[1] J. K. Kabziński, A. Wroński, *On equivalential algebras*, **Proceedings of the 1975 International Symposium of Multiple-Valued Logics**, Indiana University, Bloomington, May 13-16, 1975, pp. 419–428.

[2] R. E. Tax, *On the intuitionistic equivalential calculus*, **Notre Dame Journal of Formal Logic** 14 (1973), pp. 448–456.

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