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S. K. Thomason

POSSIBLE WORLDS AND MANY TRUTH VALUES

This is an abstract of a paper submitted to Studia Logica. The results were presented at the 1977 Spring Meeting of the Association for Symbolic Logic.

In many-valued modal logic, validity is defined with reference to frames, as in ordinary modal logic, except that valuations assign to each formula, at each possible world, not just T or F but a truth value from a fixed many-valued truth-functional logic. It is required that $\Box\alpha$ be assigned a designated truth value at a possible world if and only if is assigned designated values at all alternative worlds. Previous work has established the existence of analogues in certain many-valued modal logics of certain familiar systems of ordinary modal logic.

Theorem 1. *Any formula α of any many-valued modal logic $\underline{M}$ determines the same class of frames as some formula α^* of ordinary modal logic $\underline{K}$.*

The proof of Theorem 1 proceeds as follows. Without loss of generality, we assume that $\underline{M}$ is "standard", i.e. that $\underline{M}$ has "standard" connectives $\neg, \vee, \tau_a$ (for each truth value a) so that $\neg b,\ b \vee c, \tau_a b$ is designated iff b is not designated, at least one of b, c is designated, $b = a$, respectively. The given formula α can be written as $\beta({}^*\alpha_1 \ldots \alpha_m/p)$, where $\alpha_1, \ldots, \alpha_m$ are standard. Then α is replaced by $\gamma \Rightarrow \beta$ (i.e. by $\neg\gamma \vee \beta$), where γ has only standard connectives and $\Box$ and "says" that, necessarily, p has the same truth value as ${}^*\alpha_1 \ldots \alpha_m$ (for every assignment of truth values to $\alpha_1, \ldots, \alpha_m$). Finitely many transformations of this sort yield a formula α' having only standard connectives and $\Box$, which is valid on exactly the same frames as α. Now α' is constructed from formulas of the form $\tau_a\beta$

using only the connectives $\neg, \vee, \Box$ – if β is not a variable, then $\tau_a\beta$ can be replaced by a compound (using just $\neg, \vee, \Box$) of formulas $\tau_b\gamma$ with γ shorter than β. Finitely many replacements of this sort lead to a formula α'' having only standard connectives and $\Box$, in which τ_a's apply only to variables; and α'' is valid on the same frames as α' (in fact, $\alpha' \Leftrightarrow \alpha''$ is valid on every frame). In α'', replace every variable p not within the scope of any τ_a by the disjunction of all $\tau_b p$ with b designated; the formula α''' so obtained is constructed, via $\neg, \vee, \Box$, from formulas $\tau_{a_i} p_j$, and again $\alpha'' \Leftrightarrow \alpha'''$ is valid on every frame. Finally, let β be obtained from α''' by replacing each $\tau_{a_i} p_j$ by a new variable q_{ij}, let γ be a formula which "says" that, necessarily, for each j exactly one q_{ij} holds, and let α^* be $\gamma \Rightarrow \beta$. Then α^* is a formula of $\underline{K}$ valid on exactly the same frames as α, as required.

Formal systems in $\underline{K}$ are assumed to have just the usual rules (Substitution, Detachment, Necessitation), and at least the axioms of the system K. If $\underline{M}$ is standard, then formal systems in $\underline{M}$ are assumed to have, in addition to the usual rules, a rule called Elimination: from α''' infer α^* (where α''' and α^* are as in the preceding paragraph). Also, formal systems in $\underline{M}$ are assumed to have axioms which include a certain finite set of "$\underline{M}$-axioms"; these $\underline{M}$-axioms are valid on all frames. $K^{\underline{M}}$ is the system in $\underline{M}$ whose axioms are just the $\underline{M}$-axioms. In the remainder of this paper, $\underline{M}$ is assumed to be standard.

LEMMA 2. (Completeness theorem for $K^{\underline{M}}$). *$K^{\underline{M}} \vdash \alpha$ iff α is valid on all frames.*

The proof from left to right is trivial; the proof from right to left proceeds as follows. If α is valid on all frames, then by Theorem 1 and the Completeness theorem for K, $K \vdash \alpha^*$. Since $K^{\underline{M}}$ extends K, $K^{\underline{M}} \vdash \alpha^*$. Now we proceed along the same line as the proof of Theorem 1, only backwards, showing successively that α''', α'', α', and α are theses of $K^{\underline{M}}$. (The $\underline{M}$-axioms are chosen just so as to make this possible).

The Elimination rule is not used in the proof of Lemma 2. That proof shows, in fact, that if S is any system in $\underline{M}$ and $S \vdash \alpha^*$ without use of Elimination, then $S \vdash \alpha$ without use of Elimination.

THEOREM 3. *If S is any system in $\underline{M}$, then $S \vdash \alpha$ iff $S \vdash \alpha^*$.*

The proof from right to left is just as in Lemma 2. For the other direction, use Theorem 1, Lemma 2, and the Elimination rule to proceed step-by-step from α to α^*.

COROLLARY 4. *If S and S' are systems in $\underline{M}$, and $S \vdash \beta$ iff $S' \vdash \beta$ for all formulas β of $\underline{K}$, then S and S' are equivalent.*

COROLLARY 5. *Every system in $\underline{M}$ is axiomatizable by formulas of $\underline{K}$ (in addition to the $\underline{M}$-axioms).*

Let S now be a system in $\underline{K}$, and consider the system $S^{\underline{M}}$ in $\underline{M}$ whose axioms are just the axioms of S (together with the $\underline{M}$-axioms). It is not obvious that every formula β of $\underline{K}$ which is provable in $S^{\underline{M}}$ is provable in S. (Presumably, one would prove, by induction on proofs, that if $S^{\underline{M}} \vdash \alpha$ then $S \vdash \alpha^*$; the inductive step is not obvious). If, however, S is complete with respect to some class F of frames, then $S \nvdash \beta \Rightarrow (W \not\models \beta$ for some $W \in F)$ $\Rightarrow$ (W is a frame for $S^{\underline{M}}$ on which β is not valid) $\Rightarrow S^{\underline{M}} \nvdash \beta$. Moreover, in this case the Elimination rule is redundant in $S^{\underline{M}}$: for if $S^{\underline{M}} \vdash \alpha$ then $S \vdash \alpha^*$, but S has no Elimination rule and every proof in S is a proof in $S^{\underline{M}}$; thus $S^{\underline{M}} \vdash \alpha^*$ without Elimination, and by the remarks following the proof of Lemma 2, $S^{\underline{M}} \vdash \alpha$ without Elimination.

Thus, whenever S is a complete system of ordinary two-valued modal logic, and $\underline{M}$ is a standard many-valued modal logic, there is an analogue $S^{\underline{M}}$ of S in $\underline{M}$ such that

i. $S^{\underline{M}}$ is complete (for any class of frames for which S is complete),
ii. for formulas β of $\underline{K}$, $S^{\underline{M}} \vdash \beta$ iff $S \vdash \beta$,
iii. $S^{\underline{M}}$ is decidable iff S is decidable,
iv. $S^{\underline{M}}$ is finitely axiomatizable iff S is finitely axiomatizable.

Moreover $S^{\underline{M}}$ can be taken to have only the usual rules of inference.

Department of Mathematics
Simon Fraser University
B.C., Canada