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THE LATTICE OF MODAL LOGICS (PRELIMINARY REPORT)

A modal logic L is understood to be a set of modal formulas (built up from propositional variables and the connectives $\rightarrow, \bot, \Box$ in the usual way) containing the axiom $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ and closed under modus ponens, substitution and necessitation, i.e. if $A \in L$ then $\Box A \in L$. The smallest modal logic will be denoted by K. If L is a modal logic then L_1 is said to be an extension of L if L_1 is a modal logic such that $L \subseteq L_1$. The following extensions of K are of particular importance:

T	characterized by the axiom	$\Box p \rightarrow p$
$S4$	characterized by the axioms	$\Box p \rightarrow p,\ \Box p \rightarrow \Box\Box p$
PC	characterized by the axiom	$\Box p \leftrightarrow p$

In [1] we have studied the lattice of extensions of $S4$, using methods of an algebraic nature. These methods can be applied equally successfully in the investigation of the lattice of modal logics as a whole. The widely-used Kripke semantics proves to be too coarse a tool for a proper study of this lattice. Fine [2] has given an example of an extension of $S4$ which is not characterized bij its Kripke frames. He introduces the notion degree of incompleteness $\delta(L)$ of a logic L:

$$\delta(L) = |\{L' : L' \text{ an extension of } K \text{ such that for every Kripke frame } F, F \models L' \text{ iff } F \models L\}|$$

Theorem 1. $\delta(PC) = 2^{\aleph_0}$.

In fact, it follows from the construction of our example that

Corollary 2. *Let L be an extension of K containing $\Box^n p \leftrightarrow \Box^{n+1} p$ for some natural number n or $\Box p \to p$. Then $\delta(L) = 2^{\aleph_0}$.*

We have not been able to prove that every proper extension of K has degree of incompleteness $2^{\aleph_0}$, as yet, though the conjecture seems plausible.

It is well-known that the lattice of modal logics is distributive and has the cardinality of the continuum. A modal logic L_1 is called an immediate predecessor of L_2 if $L_1 \subsetneq L_2$ and for every modal logic L such that $L_1 \subseteq L \subseteq L_2$ either $L_1 = L$ or $L = L_2$. It is not difficult to show that every proper extension of K has an immediate predecessor. In [1] we gave an example of an extension of $S4$ having $\aleph_0$ immediate predecessors which are also extensions of $S4$.

Theorem 3. *There is an extension of K having $2^{\aleph_0}$ immediate predecessors.*

This result shows that if we wish to represent the lattice of modal logics as a lattice of subsets of a set X, this set X should have the cardinality of the continuum.

More generally, we may ask if for extensions L_1, L_2 of K such that $L_1 \subsetneq L_2$ there always an immediate predecessor of L_2 which is an extension of L_1. If L_1 has the finite model property or L_2 is finitely axiomatizable this is easily seen to be true. However

Theorem 4. *There exist extensions L_1, L_2 of K such that $L_1 \subsetneq L_2$ and L_2 does not have an immediate predecessor which is an extension of L_1.*

A modal logic is called finite if it is the set of modal formulas satisfied bij some finite Kripke frame. In [1] the finite extensions of $S4$ have been characterized as the extensions of $S4$ which have only finitely many extensions. We also show there that every finite extensions of $S4$ has finitely many immediate predecessors which are extensions of $S4$, all of which are finite. As regards extensions of T, the situation is radically different:

Theorem 5. *There are infinitely many finite extensions of T which are immediate predecessors of PC.*

This result was found independently by W. Rautenberg (cf. [3]).

Theorem 6. *There is a non-finite extension of T which is an immediate predecessor of PC.*

This example refutes the conjecture in [3].

References

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[3] W. Rautenberg, *Some properties of the hierarchy of modal logics*, **Bulletin of the Section of Logic** 5 (1976), no. 3, pp. 103–106.

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