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ON FREE RELATIVELY PSEUDOCOMPLEMENTED SEMILATTICE WITH THREE GENERATORS

The full text of this paper will appear in Report on Mathematical Logic.

Relatively pseudocomplemented semilattice (alias implicative semilattice) is a meet semilattice $\langle A, \leqslant \rangle$ s.t. for every $a, b \in A$ there exists a relative pseudocomplement of a w.r.t. b, i.e. the greatest element in the set $\{x \in A : inf(a, x) \leqslant b\}$. Implicative semilattices form a variety, when they are treated as algebras with two binary operations $\wedge$ and $\rightarrow$. Let us denote this variety by $\mathbb{I}$. $\mathcal{T}_n$ will denote the n-freely-generated free algebra in $\mathbb{I}$. It is known that all the algebras $\mathcal{T}_n$, $n = 1, 2, \ldots$ are finite and thus the question arises the number of elements of $\mathcal{T}_n$. Balbes in [1] proved that $\mathcal{T}_2$ possesses 18 elements and asked for the number of elements of $\mathcal{T}_3$. In this paper we will answer Balbes' question.

Lemma 1. (Balbes, Dwinger [2]) *Let $\mathcal{F}$ be a free algebra of type τ and let $\mathcal{A}$, $\mathcal{L}$ be algebras of type τ. Then for every epimorphisms $f : \mathcal{F} \mapsto \mathcal{A}$, $g : \mathcal{F} \mapsto \mathcal{L}$ the following conditions are equivalent:*

- *(i) $Ker(f) \subseteq Ker(g)$,*
- *(ii) there exists an epimorphism $h : \mathcal{A} \mapsto \mathcal{L}$ such that $h * f = g$.*

If $\mathcal{A}$ is a meet semilattice then $\mathbb{C}(\mathcal{A})$, $\mathbb{F}(\mathcal{A})$ denote the lattice of all the congruence relations of $\mathcal{A}$ and the lattice of all the filters of $\mathcal{A}$ respectively. If $\mathcal{A}$ is a lattice then the symbols $\nabla(\mathcal{A})$ $(\Delta(\mathcal{A}))$ denote the set of all the sup-irreducible (inf-irreducible) elements of $\mathcal{A}$. The set $X \subseteq \nabla(\mathcal{A})$ is hereditary iff $a \leqslant b \in X$ implies that $a \in X$. The family of all the hereditary subsets of $\nabla(\mathcal{A})$ ordered by the inclusion will be denoted by $H(\nabla(\mathcal{A}))$.

THEOREM 1. (Birkhoff [3]) *Every finite distributive lattice $\mathcal{A}$ is isomorphic with $H(\nabla(\mathcal{A}))$.*

COROLLARY. *If $\mathcal{A}$ is finite distributive lattice then $\Delta(F(\mathcal{A}))$ is isomorphic with $\nabla(\mathcal{A})$.*

THEOREM 2. *If $\mathcal{A} \in \mathbb{I}$ then the lattices $\mathbb{C}(\mathcal{A})$ and $\mathbb{F}(\mathcal{A})$ are isomorphic.*

LEMMA 2. *For every $\mathcal{A} \in \mathbb{I}$ the following conditions are equivalent:*

(i) $\mathcal{A}$ is subdirectly indecomposable,

(ii) there exists the greatest of all non-unit elements of $\mathcal{A}$ (we denote it by $*$).

LEMMA 3. (Wroński [9]) *If $\mathcal{A}$ is a subdirectly indecomposable implicative semilattice then the following conditions hold:*

(i) $\mathcal{A}' = \langle A - \{*\}, \wedge, \rightarrow \langle$ is a subalgebra of $\mathcal{A}$,

(ii) if $G \subseteq A$ generates $\mathcal{A}$ then $G - \{*\}$ generates $\mathcal{A}'$.

THEOREM 3. (Diego [5], Popiel (see Jankov [7], p. 24)) *Algebras $\mathcal{T}_n$ are finite, $n = 1, 2, \ldots$.*

Using Lemma 3 and the description of $\mathcal{T}_2$ given by Balbes [1] it is easy to verify that:

LEMMA 4. *The algebras $\vartheta_1, \ldots, \vartheta_9$ (see fig. 1) are all (up to isomorphism) subdirectly indecomposable homomorphic images of $\mathcal{F}_3$.*

Given and implicative semilattice $\mathcal{A}$, the epimorphisms from $\mathcal{T}_3$ onto $\mathcal{A}$ are in one-one correspondence with three-termed sequences $\langle a_1, a_2, a_3\rangle$ of elements of $\mathcal{A}$ s.t. the set $\{a_1, a_2, a_3\}$ generates $\mathcal{A}$. Such sequences will be referred to as $\mathcal{A}$-sequences. Given two ϑ_i-sequences s_0 and s_1 we say that they are equivalent iff $Ker(\varphi_0) = Ker(\varphi_1)$ where φ_0 and φ_1 are corresponding epimorphisms of $\mathcal{T}_3$ onto ϑ_i. For $i = 1, \ldots, 9$ let S_i be the set of all equivalence classes of ϑ_i-sequences and let $S = S_1 \cup \ldots \cup S_9$. We define a partial ordering of the set s putting for every $s_0, s_1 \in S$, $s_0 \leqslant s_1$ iff $Ker(\varphi_0) \subseteq Ker(\varphi_1)$ where φ_0, φ_1 are corresponding epimorphisms from $\mathcal{T}_3$. We can use now Lemma 1 and by inspection of possible epimorphisms from ϑ_i onto ϑ_j, $i \neq j$, we obtain the diagram of $\mathcal{J} = \langle S, \leqslant\rangle$ shown in fig. 2 (the order is decreasing as we move from the center of the diagram, the only separated point is placed in the center).

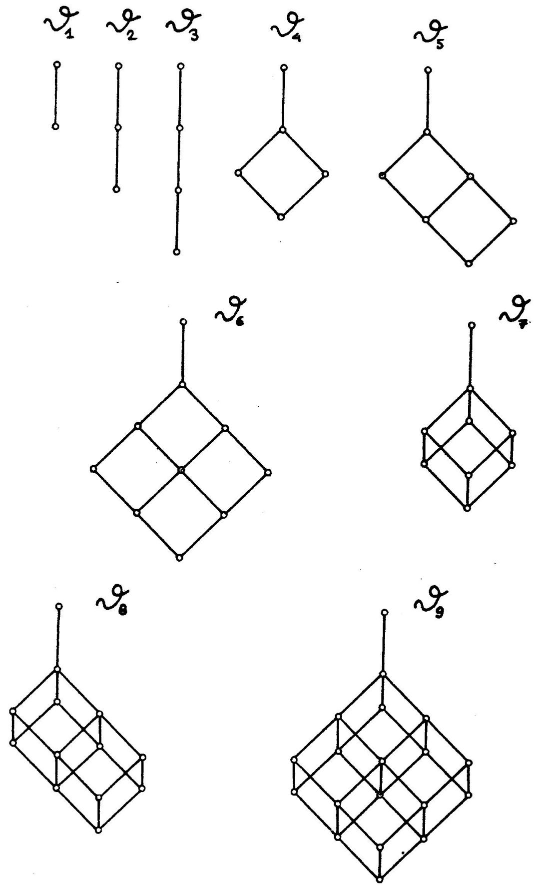

FIGURE 1

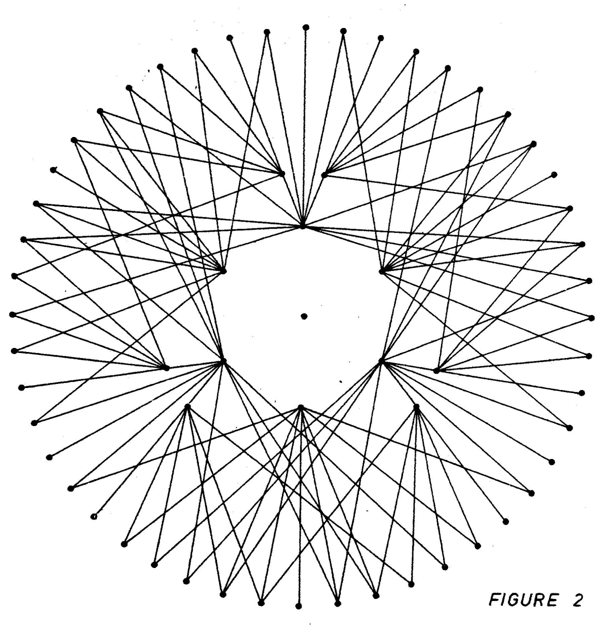

FIGURE 2

THEOREM 4. *The free implicative semilattice $\mathcal{T}_3$ is order-isomorphic to the partially ordered set $\langle H(\mathcal{J}), \subseteq \rangle$.*

COROLLARY. *The number of elements of $\mathcal{T}_3$ is equal to the number of antichains of $\mathcal{J}$, i.e. 623 662 965 552 330.*

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