Bulletin of the **S**ection of **L**ogic
Volume 6/4 (1977), pp. 151–154
reedition 2011 [original edition, pp. 151–155]

Roman Suszko

ON FILTERS AND CLOSURE SYSTEMS

(Presented to the Logic Seminar, Wrocław, 6 June 1977)

This report brings out a simple observation on the close connection of filters with algebraic closure systems. In [1], Orrin Frink gave a general definition of ideals in ordered sets. Here, we use the dual notion of filter and apply it to preordered sets.

When referring to finite sets $\{c_1, \ldots, c_k\}$ we often omit the brackets. The symbol $\emptyset$ denotes the empty set and, $X \subseteq_f Y$ means that X is a finite subset of Y.

Frink's filters in preordered sets

Let the relation $\vdash$ be reflexive and transitive on the nonempty set A. Then, the pair $\underline{A} = \langle A, \vdash \rangle$ is called a preordered set. For any $a, b, c \in A$ and $Z \subseteq A$, assume four definitions:

$a \vdash\dashv b$ iff $a \vdash b$ and $b \vdash a$
$[c) = \{z \in A : c \vdash z\}$
$(c] = \{z \in A : z \vdash c\}$
$Z^+ = \bigcap\{[b) : b \in Z\}$
$Z^\circ = \bigcap\{(a] : a \in Z\}$

All the sets $[c)$ for $c \in A$ are called principal filters of $\underline{A}$.

The operation $Z \mapsto Z^{\circ+}$ appears to be a closure operator on the set A. The corresponding family of closed sets, $\underline{D}$, is the smallest closure system on A containing all principal filters of $\underline{A}$. Notice that $\emptyset^{\circ+} = A^+, c^{\circ+} = [c)$ and, $d \in \{c_1, \ldots, c_k\}^{\circ+}$ iff for all $z \in A$, $z \vdash d$ whenever $z \vdash c_1, \ldots, z \vdash c_k$.

For any $Z \subseteq A$ put $Inf(Z) = Z^\circ \cap Z^{\circ+}$. Hence, $Inf(\emptyset) = A^+$ and, $d \in Inf(c_1, \dots, c_k)$ if and only if two conditions hold:

(1) $d \vdash c_1, \dots, d \vdash c_k$
(2) for all $z \in A$, $z \vdash d$ if $z \vdash c_1, \dots, z \vdash c_k$.

Observe that $a \dashv\vdash b$ whenever $a, b \in Inf(Z)$. Also notice that if $Inf(a, b) \neq \emptyset$ for all $a, b \in A$ then for any $c_1, \dots, c_k \in A$, $Inf(c_1, \dots, c_k) \neq \emptyset$.

A set $F \subseteq A$ is called a filter of $\underline{A}$, $F \in \underline{F}$, if $A^+ \subseteq F$ and for any $c_1, \dots, c_k \in F$, $\{c_1, \dots, c_k\}^{\circ+} \subseteq F$. It has been shown in [2] that the family $\underline{F}$ of all filters of $\underline{A}$ is the smallest algebraic closure system on A containing all principal filters. In other words, $\underline{F}$ is exactly the family of all unions of directed families of arbitrary intersections of principal filters.

For any $Z \subseteq A$ put $\widetilde{Z} = \bigcup\{Y^{\circ+} \subseteq A : Y \subseteq_f Z\}$. It is easy to see that the operation $Z \mapsto \widetilde{Z}$ is the closure operator corresponding to $\underline{F}$. Furthermore, $\underline{D} \subseteq \underline{F}$ and $\widetilde{Z} \subseteq Z^{\circ+}$ for each $Z \subseteq A$. Moreover, $\widetilde{Z} = Z^{\circ+}$ whenever $Z \subseteq_f A$. In particular, $\widetilde{\emptyset} = A^+$ and $\widetilde{c} = [c)$ for every $c \in A$.

For any filter F of $\underline{A}$ the following three conditions hold:

(3) $Inf(\emptyset) \subseteq F$
(4) $Inf(c_1, \dots, c_k) \subseteq F$ whenever $\{c_1, \dots, c_k\} \subseteq F$
(5) $[c) \subseteq F$ whenever $c \in F$.

However, in case $Inf(a, b) \neq \emptyset$ for all $a, b \in A$, these properties characteristic precisely filters of $\underline{A}$, i.e., (3), (4), (5) imply that F is a filter.

2. Filters in closure systems

Let $\underline{C}$ be a closure system on $A \neq \emptyset$ and let $Z \mapsto \overline{Z}$ be the corresponding closure operator on A. Write $a \vdash b$ for $b \in \overline{a}$ whenever $a, b \in A$. Then, $\underline{A} = \langle A, \vdash \rangle$ is a preordered set, indeed. We are going to relate $\underline{C}$ to the family $\underline{F}$ of all filters of $\underline{A}$. Clearly, $\widetilde{c} = \overline{c}$ for each $c \in A$ and $\widetilde{\emptyset} = \bigcap\{\overline{z} \subseteq A : z \in A\}$. Also, $\overline{\emptyset} \subseteq \widetilde{\emptyset}$ and $\overline{c_1, \dots, c_k} \subseteq \widetilde{c_1, \dots, c_k}$ for any $c_1, \dots, c_k$ in A. Consequently,

(I) if $\underline{C}$ is algebraic then $\underline{F} \subseteq \underline{C}$.

It is easy to prove the lemma:

(II) if $e \in Inf(c_1, \dots, c_k)$ then

$$c_1, \dots, c_k \in \overline{Inf(c_1, \dots, c_k)} = \overline{e}.$$

The question when $\underline{C} = \underline{F}$ is answered by the following equivalence:

(III) $\underline{C} = \underline{F}$ if and only if the following three conditions hold:
(1) $\underline{C}$ is algebraic,
(2) $\bigcap\{\overline{z} \subseteq A : z \in A\} \subseteq \overline{\emptyset}$,
(3) for all $c_1, \dots, c_k, d \in A, d \in \overline{c_1, \dots, c_k}$
whenever for all $z \in A$, if $c_1, \dots, c_k \in \overline{z}$ then $d \in \overline{z}$.

For the proof, observe that (2) and (3) together mean that $\widetilde{Z} \subseteq \overline{Z}$ for all $Z \subseteq_f A$. Indeed, this is a separation property (filter principle): the family of all principal filters $\overline{c}$, $c \in A$, separates closures of finite sets (finitely axiomatizable theories) from points (formulas).

Assume that $\underline{C}$ is algebraic and such that $Inf(a, b) \neq \emptyset$, for all $a, b \in A$. Then,

(IV) $\underline{C} = \underline{F}$ if and only if (4) $Inf(\emptyset) \subseteq \overline{\emptyset}$ and
(5) $Inf(a, b) \subseteq \overline{a, b}$ for all $a, b \in A$.

Proof. One way is easy. For the other way, it is enough to show that (5) entails (3). Since $\emptyset \neq Inf(a, b) \subseteq \overline{a, b}$ for all $a, b \in A$ it follows that for any $c_1, \dots, c_k \in A$, $\emptyset \neq Inf(c_1, \dots, c_k) \subseteq \overline{c_1, \dots, c_k}$. To show (3), assume that for each $z \in A$, $d \in \overline{z}$ whenever $c_1, \dots, c_k \in \overline{z}$ and, suppose, $e \in Inf(c_1, \dots, c_k)$. Then, $c_1, \dots, c_k \in \overline{Inf(c_1, \dots, c_k)} = \overline{e}$ by the lemma (II). Hence, $d \in \overline{e}$. But $\overline{Inf(c_1, \dots, c_k)} \subseteq \overline{c_1, \dots, c_k}$ and, consequently, $e \in \overline{c_1, \dots, c_k}$. Thus, $d \in \overline{c_1, \dots, c_k}$.

Suppose that the set A is endowed with an operation $a, b \mapsto a \cdot b$. Then, the closure system $\underline{C}$ is called conjunctive if $\overline{a \cdot b} = \overline{a, b}$ for all $a, b \in A$. In this case, one may easily show that the said binary operation is associative ($mod.$ $\vdash\dashv$) and $\overline{c_1 \cdot \ldots \cdot c_k} = \overline{c_1, \dots, c_k}$ for any $c_1, \dots, c_k \in A$. Obviously, we have the lemma:

(V) if $\underline{C}$ is conjunctive then for all $a, b \in A$,

$$a \cdot b \in Inf(a, b) \subseteq \overline{a, b}.$$

It follows that if $\underline{C}$ is algebraic and conjunctive then

(VI) $\underline{C} = \underline{F}$ iff $\bigcap\{\overline{z} \subseteq A : z \in A\} \subseteq \overline{\emptyset}$.

References

[1] O. Frink, *Ideals in partially ordered sets*, **Amer. Math. Monthly** 61 (1954), pp. 223–234.

[2] J. Mayer-Kalkschmidt and E. Steiner, *Some theorems in set theory and applications in the ideal theory of partially ordered sets*, **Duke Math. Journal** 31 (1964), pp. 287–289.

The Section of Logic
Institute of Philosophy and Sociology
Polish Academy of Sciences