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THE LATTICE OF NORMAL MODAL LOGICS (PRELIMINARY REPORT)

Most material below is ranked around the splittings of lattices of normal modal logics. These splittings are generated by finite subdirect irreducible modal algebras. The actual computation of the splittings is often a rather delicate task. Refined model structures are very useful to this purpose, as well as they are in many other respects. E.g. the analysis of various lattices of extensions, like $ES5$, $ES4.3$ etc becomes rather simple, if refined structures are used. But this point will not be touched here.

The variety TBA (cf. below), which corresponds to $S4$, is congruence-distributive. Hence any every tabular extension $L \supseteq S4$ has finite many extensions only. Around 1975 it has been proved by several authors, that also the converse is true: If $L \supseteq S4$ has finitely many extensions only, then L is necessarely tabular. These and other results will be extended to much richer lattices. For simplicity we state the results explicitely only for the lattice N of modal logics with one modal operator. But most of them carry over literally to the lattice N^k of k-ramified normal modal logics (k modal operators). It is easy to construct examples of 2-ramified modal logics (tense logics) which are $POST$-complete but not tabular. Whether this will be possible for normal (unramified) modal logic seems to be an open problem. In view of results below it seems very unlikely that such examples exist.

0. Notation

$V = \{p_i | i \in \omega\}$ = set of (propositional) variables. $\text{Ł} = \{P, Q, \ldots\}$ = set of formulas built up from V and the connectives $\neg, \wedge, \vee, \rightarrow, \Box$ ($\Diamond$ is introduced as usual). $\Box^n p := \underbrace{\Box\ldots\Box}_{n} p$; $np := p \wedge \Box p \wedge \ldots \Box^n p$ $(n \in \omega)$. N denotes the lattice of all normal modal logics, the smallest element of N is denoted by K, the largest is the inconsistent Ł. Put $EL := \{L' \in N / L' \supseteq L\}$. $K(P)_{P\in X} := KX :=$ smallest $L \in N$ containing $X \subseteq$ Ł. $K(P) := K(\{P\})$. $K^n := K(np \to \Box^{n+1}p)$ is the *n-transitive* logic $(n \in \omega)$.

KBA denotes the variety of K-Algebras (= normal modal algebras), which are Boolean algebras A with an additive unary operator $\blacksquare : A \to A$ with some well-known properties. $\blacksquare^n a := \blacksquare\ldots\blacksquare a$; $na := a \wedge \blacksquare a \wedge \ldots \blacksquare^n a$ $(a \in A)$. If $\mathbf{C} \subseteq KBA$, $\mathbf{C}_{s.i.} :=$ class of all subdirect irreducible members of $\mathbf{C}$. KBA_f = class of finite $A \in KBA$. $LA :=$ *logic of* $A \in KBA$ in the usual sense. Always $LA \in N$, and each $L \in N$ has a representation $L = LA$, some $A \in KBA$. $L\mathbf{C} := \bigcap_{A\in\mathbf{C}} LA$ for $\mathbf{C} \subseteq KBA$.

$MdL := \{A \in KBA | L \subseteq LA\}$ for $L \in N$, and $Md_{s.i.}L := MdL \cap KB_{s.i.}$. As is well-known $L = LMdL = LMd_{s.i.}L$. $L \in N$ is said to be *tabular*, if $L = LA$, some $A \in KBA_f$. $L \in N$ is *pretabular* if L itself is not, but each proper extension $L' \in EL$ is tabular. $F\ell A :=$ lattice of congruence filters of $A \in KBA$. $F\ell A \simeq C\ell A$ (:= congruence lattice), and $F\ell A$ turns out to be a sublattice if $F\ell A^0$, where A^0 is the Boolean reduct of A. Hence KBA is congruence-distributive, and Jónsson's theory on congruence-distributive varieties does apply. In particular (i) N is distributive, (ii) $Md_{s.i.}(L_1 \sqcap L_2) = Md_{s.i.}L_1 \cup Md_{s.i.}L_2$ and (iii) If $L \supseteq LA$ $(A \in KBA_f)$ then $Md_{s.i.}L' \subseteq HSA$. Note that KBA has the Congruence Extension Property. Hence $HSA = SHA$ (S, H operators of taking substructures and homomorphisms, resp.) $\mathbf{G}$ denotes the class of *K-frames*; each $g \in \mathbf{G}$ being a set $\{S, T \ldots\}$ with a binary relation generally denoted by $\lhd$. $S \rhd T := T \lhd S$. $S \unlhd T := S \lhd T$ or $S = T$. If $a \subseteq g$, then $\rhd^0\ a := a$; $\rhd^{n+1}\ a := \{S' \in g / S \lhd S'$, some $S \in \rhd^n\ a\}$; $\rhd^n\ S := \rhd^n\ \{S\}$. $\delta a := \bigcup_{n\in\omega} \rhd^n\ a$. g is *initial*, if $g = \delta S$, some $S \in g$ $(\delta S := \delta\{S\})$. If $\delta S = \delta S'$, then S, S' are said to be *equilocal*. The equivalence classes of this relation are called *clusters* of g. These form in an obvious sense a poset g^*. g itself is a poset iff $g = g^*$. Poset g is of *progression order* $n \geqslant 1$, if there is a chain of length in g, but not of length $n + 1$. An arbitrary $g \in G$ is of progressive order n, if g^* is of progressive order n. Symmetric relations e.g. are of progressive order 1.

A^+g denotes the Boolean algebra of subsets of g expanded by the unary operation $\blacksquare : 2^g \to 2^g$ such that $\blacksquare a = \{S' \in g | \rhd S \subseteq a\}$. Obviously

$A^+g \in KBA$. Put $K^mBA := MdK^m$. One easily shows that $A^+g \in K^mBA$ if $\delta S = \bigcup_{i \leqslant m} \rhd^i S$ $(S \in g)$. In particular, $A^+g \in K^1BA$ iff $S_0 \lhd S_1 \lhd S_2 \Rightarrow S_0 \unlhd S_2$ $(S_0, S_1, S_2 \in g)$. Thus, 1-transitivity corresponds to a weak form of the customary transitivity. Lg = modal logic of g in the sense of relational semantics. $L = Lg = LA^+g$ is easily proved. $LK := \bigcap_{g \in K} Lg$ for $K \subseteq G$. $MsL := \{g \in G | L \subseteq Lg\}$ is the class of L-structures. L is said to be *complete* (with respect to relational semantics) if $L = Lg$, some $g \in G$. Each complete $L \in N$ has a representation $L = LK$, K some set of initial $g \in G$. A pair $\gamma = (g, B)$, where A is a subalgebra of A^+g, is called a *generalized frame*. γ is *refined* if (i) $S \neq S' \Rightarrow (\exists a \in A)(S \in a \wedge S' \notin a\}$ and (ii) $S \not\lhd S' \Rightarrow (\exists a \in A)(S \in \blacksquare a \wedge S' \notin \blacksquare a)$. Each $L \in N$ has a representation $L = L\Gamma$, Γ some set of initial refined $\gamma = (g, A)$. If $L \supseteq S4$, one may assume in addition that the initial cluster of g belongs to A in each of the $(g, A) \in \Gamma$. MgL denotes the class of all initial refined γ such that $L \subseteq L\gamma$.

1. Some general properties of N

We start with some general observations. As we have stated before, N is distributive. But we may achieve something more. First the following subdirect irreducibility criterion is easily proved. Well-known criteria e.g. for $A \in TBA_{s.i.}(TBA = MdS4)$ derive from it as special cases.

Proposition 1. *$A \in KBA_{s.i.}$ iff $\exists_{d \neq 1} d \in A : \forall_{a \neq 1} a \in A : \exists n \in \omega : na \leqslant d$.*

From this the following disjunction property is easily obtained: If $A \in KBA_{s.i.}$ then $(\forall n \in \omega)$ $(na \cup nb = 1)$ implies $a = 1$ or $b = 1$. If $A \in K^mBA$, the elements $ma (a \in A)$ are said to be *open*. The operator m behaves in $A \in K^mBA$ like $\blacksquare$ in $A \in TBA$. Since $na = ma$ for $n \geqslant m$, we get

Proposition 2. *$A \in K^mBA_{s.i.}$ iff A contains a largest open element $\neq 1$.*

Therefore $A \in K^mBA$ has the following disjunction property: If $ma \cup mb = 1$, then $a = 1$ or $b = 1$.

$P \vee Q$ denotes $P \vee Q'$, where Q' is the result of replacing the variables in Q in such a way that P, Q' have no common variables.

PROPOSITION 3. $KX \sqcap KY = K(nP \vee nQ)_{P \in Q; Q \in Y; n \in \omega}$.

The proof follows easily from the fact that $L \in N$ is determined by $Md_{s.i.}L$, and from the disjunction property.

PROPOSITION 4. *N is a Heyting-Algebra (in particular, N is distributive).*

PROOF. Let $L_i = KX_i$ $(i = 1, 2)$. Put $Z = \{Q \in Ł | nP \vee nQ \in L_2$ for all $P \in X$ and all $n \in \omega\}$. It is easily proved that $L_0 : KZ$ is the largest $L \in N$ such that $L_1 \cap L \subseteq L_2$.

Another consequence from Prop. 3 is the fact that the finite axiomatizable $L \supseteq K^m$ form a sublattice of EK^m. This is not the case for the whole N.

Let L^p denote the positive universal fragment of the first order language for KBA (with predicates $=, \leqslant$, and the operation in KBA), i.e. negation, implication and existential quantification do not occur in the formulas $\varphi \in L^p$.

PROPOSITION 5. *For each $m \geqslant 1$ there is a translation $\tau_m : L^p \to Ł$ such that $\models^A \varphi \Leftrightarrow \varphi^{\tau_m} \in LA$ for each $A \in K^m BA_{s.i.}$.*

τ_m is explained by example. If e.g. $\varphi = \bigvee_{i<j<2^n} x_i = x_j$, then $\varphi^{\tau_m} = \bigvee_{i<j\leqslant 2^n} m(p_i \leftrightarrow p_j)$, and φ, φ^{τ_m} then both express the fact that $A \in K^m BA_{s.i.}$ contains at most 2^n elements. This implies

PROPOSITION 6. *$L \in N$ is tabular iff $\bigvee_{i<j<2^m} m(p_i \leftrightarrow p_j) \because mp_0 \to \Box^{m+1} p_0$ for some $m \in \omega$.*

COROLLARY. *The tabular $L \in N$ form a filter $N\tau$ in N. Each $L \in N\tau$ has finitely many extensions only.*

The figure at the end shows the upper part of $ES4$ and the upper part of N (including but some of the logics of the lattice theoretic dimension 2). That e.g. the cyclic logics Lk_p (p prime) are of dimension 2 is a consequence of the fact mentioned above, that $LMd_{s.i.}L \subseteq HSk_p$ for $L \supseteq Lk_n$. For it can easily be seen that for tabular $L \in N$ the following conditions are equivalent, and hence $A^+k_p \in K^pBA_{s.i.}$:

(i) $L = LA$, some $A \in KBA_{f.s.i.}$

(ii) $L = Lg$, some finite initial $g \in G$

(iii) L is prime ($\sqcap$-irreducible) in the lattice $N\tau$ of tabular $L \in N$.

Another corollary from Prop. 6 is that each $L \in N$ is contained in some pretabular $L \in N$. The problem mentioned in Sec. 0 thus amounts to the question, as to whether all pretabular $L \in N$ lay "behind" the tabular $L \in N$, i.e. whether the pretabular $L \in N$ have no finite dimension in N.

2. The splitting-theorem for K^m

$A \in MdL^0$ ($L^0 \in N$) is said to *generate a splitting of* EL^0, if there is some $L^* \supseteq L^0$ such that either $L \subseteq LA$ or else $L^* \subseteq L$ for every $L \in EL^0$. L^* is uniquely determined and will be denoted by L^0/A.

SPLITTING-THEOREM. *Let $L^0 \supseteq K^m$ for some $m \in \omega$. Then each $C \in Md_{f.s.i.}L^0$ generates a splitting of L^0. Each of the following conditions is sufficient for $L^* = L^0/C$:*

(i) $L^* = L^0(P)$ and $P \notin LC$ and $C \in SHA \Rightarrow P \in LA$ for each $A \in MdL^0$

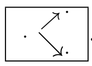
(ii) $L \not\subseteq LC$ and if $L \not\subseteq L\gamma$, $\gamma \in MgL^0$, then $L\gamma \subseteq LC$.

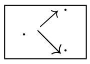
EXAMPLE. Put $C = A^+ \boxed{\cdot \to \cdot}$. We claim $S4/C = S5$. Clearly $S5 \not\subseteq LC$. Now, if $\gamma = (g, , A) \in MgS$, then $\ell_1 := \boxed{\dot{S_0} \overset{\to}{} \dot{S_1}}$ must be a substructure of g, where S_0 is initial cluster of g onto S_0, the rest onto S_1. Then not merely $Lg \subseteq L\ell_1$ but $L\gamma \subseteq L\ell_1$ (since γ is refined).

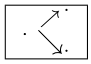
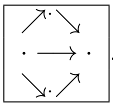
From the example the following nice Jankov-style criterion derives:

CRITERION 1. $S4(P) = S5$ *iff* $P \in S5$ and $P \notin L \boxed{\cdot \to \cdot}$.

PROOF. $\Rightarrow$: obvious. $\Leftarrow$: If $P \notin L \boxed{\cdot \to \cdot}$, then by the splitting-theorem $LP \supseteq S4/A^+ \boxed{\cdot \to \cdot} = S5$. Since $P \in S5$, $S4(P) = S5$.

CRITERION 2. $S4(P) = S4.\ 2$ iff $P \in S4.2$ and $P \notin L$. .

For it is not difficult to prove that $S4/A^+$  $= S4.2$.

CRITERION 3. $S4(P) = S4.3$ iff $P \in S4.3$ and $P \notin L$  and $P \notin L$ .

Many other criterions of this kind may be derived, refering also to other basic logics, e.g. to $K4$, and more general to any L, such that $L \supseteq K^m$ for some $m \in \omega$. A more intrisic application of the splitting-theorem is given in Sec. 3.

3. Only tabular $L \supseteq K^m$ are of finite dimension

$L \in N$ is said to be *locally finite*, if each finite generated $A \in MdL$ is finite. It is well-known that $S5$ is locally finite. This has been generalized by Blok, who showed that $S4/A^+\ell_n$ is locally finite, where ℓ_n is the linear order of $n+1$ nodes. One easily proves the

LEMMA. *If $L \in N$ is not tabular and locally finite, then L has infinitely many tabular extensions.*

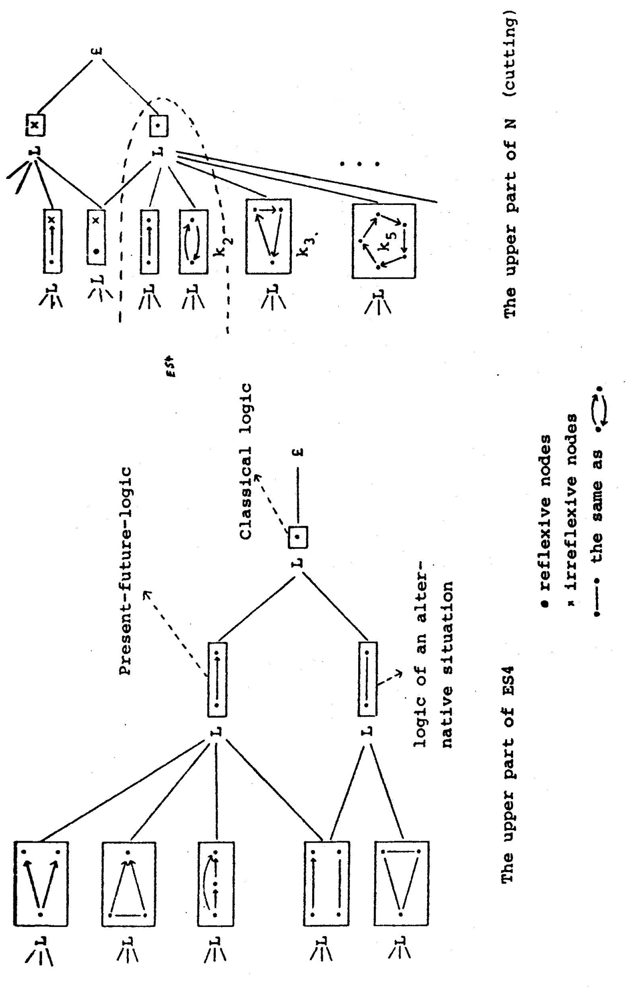
From this already follows that a non-tabular $L \supseteq S4$ has infinitely many tabular extensions. For either $L \subseteq L\ell_n$ for all $n \in \omega$ (and hence L has the different extension $L\ell_n$), or $L \supseteq L/A^+\ell_n$ for some $n \in \omega$. According to the lemma, also in this case L has infinitely many tabular extensions. The same idea is used in the proof of the theorem below. Although each non-tabular $L \in N$ is contained in some pretabular $L' \in N$, this does not imply that *any* $L \in N$ has infinite many extensions.

THEOREM. *If $L \supseteq K^m$ for some m, and L has finitely many tabular extensions only, then L itself is tabular.*

Let C_n^m denote all $C \in MdK^m$ such that $C = A^+g$ and $g^* = \ell_n$. $K^m/C_n^m := (\dots(K^m/C_1)/\dots/C_k$ where $C_n^m = (C_1, \dots, C_k)$. It can be shown that

(i) $K^m/C_n^m = LK$, where K is the class of all $g \in MsK^m$ of progressions $< n$.

(ii) K^m/C_n^m is locally finite. As a byproduct it turns out that K^m has finite many pretabular extensions only, whose number rapidly grows with n. E.g. $K4 \supseteq K^1$ has already about 25 pretabular extensions.



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