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George Epstein

# ON THE EQUIVALENCE OF THE MESKHI AND CIGNOLI CONDITIONS FOR $P$-ALGEBRAS WITH INVOLUTION, WITH APPLICATION TO ŁUKASIEWICZ 3 AND 4 VALUED LOGICS

In a recent issue of this Bulletin, S. Meskhi cites 7 additional conditions for Heyting algebras with involution and linearly ordered matrix [10, p. 11]. In [2], R. Cignoli indicates 3 additional conditions for $P$-algebras [5] with normal involution [9]. The equivalence of these conditions is shown.

The corresponding logical operator for this involution $\beta(x)$, (see [3, p. 304 and 6, pp. 76–77]) is Łukasiewicz negation, $Nx$. However, the presence of this operation need not guarantee an algebraic operation $\gamma(x, w)$ corresponding to Łukasiewicz implication $Cxw$. Referring to the finitely limited $P$-algebras of [4] as $P$-algebras of order $n$ [see also 5, pp. 205–6], it is shown that

(i) Every $P$-algebra of order 3 admits the algebraic operation $\gamma(x, w)$ and hence the operation $\beta(x)$, through $\beta(x) = \gamma(x, 0)$

(ii) Every $P$-algebra of order 4 with the Meskhi or Cignoli conditions for involution $\beta(x)$ admits the algebraic operation $\gamma(x, w)$.

The connection between 3-valued Łukasiewicz logic and Heyting algebra is not new (see [8, 12]). The unary operations discussed here also occur within the algebras of Moisil [11]. The inability of these algebras to model Łukasiewicz logics with 5 or more values was already noted by Alan Rose (see [1]). In the 4-valued case, the algebras of Moisil require four different unary operators, whereas $P$-algebras with involution require only three.

These results may be viewed easily in the interval $[0, 1]$, proceeding along the lines of [9]:

$$w \vee x = \quad max(w,x) \qquad\qquad w \wedge x = \quad min(w,x)$$

$$x \to w+ \quad \begin{cases} 1 & \text{when} \quad x \leqslant w \\ w & \text{when} \quad x > w \end{cases} \qquad \neg x = \quad \begin{cases} 1 & \text{when} \quad x = 0 \\ 0 & \text{when} \quad x > 0 \end{cases}$$

The following are drawn from [5, 6, 7]:

$$y \Rightarrow x = \quad \begin{cases} 1 & \text{when} \quad y \leqslant x \\ 0 & \text{when} \quad y > x \end{cases} \qquad !x = \quad \begin{cases} 1 & \text{when} \quad x = 1 \\ 0 & \text{when} \quad x < 1 \end{cases}$$

$$\beta(x) = 1 - x$$

As already noted, the logical operation corresponding to $\beta(x)$ is denoted in prefix form by $Nx$. In [7], $\neg$ corresponds to the logical symbol $\sim$ and ! corresponds to the logical symbol $\Box$.

The Cignoli conditions [2] that $\beta$ be a normal involution in a $P$-algebra are:

(1) $\beta(\beta(x)) = x$
(2) $w \vee x = \beta(\beta(w) \wedge \beta(x))$
(3) $\beta(!x) = \neg(!x)$

Four of the seven Meskhi conditions [10], in terms of the operations $\vee, \wedge, \neg, \beta$, may be found in [5, pp. 209–210], noting that $!x = \neg(\beta(x))$, the greatest complemented element which is $\leqslant x$. Note also the equivalence of $!(x \to w) \leqslant !x \to !w$ with $\neg(\beta(x \to w)) \leqslant \beta(w) \to \beta(x)$. Since $\beta(w) \to \beta(x) \leqslant \neg(\beta(x)) \to \neg(\beta(w))$, it follows that $!(x \to w) \leqslant !x \to !w$. Conversely, $\neg(\beta(x \to w))$ is the largest complemented element satisfying $x[\neg(\beta(x \to w))] \leqslant w$, so that $\beta(x) \vee \beta[\neg(\beta(x \to w))] \geqslant \beta(w)$ and thus $\beta(w)\neg(\beta(x \to w)) \leqslant \beta(x)\neg(\beta(x \to w)) \leqslant \beta(x)$ yielding $\neg(\beta(x \to w)) \leqslant \beta(w) \to \beta(x)$. The additional three conditions are:

(1′) $\beta(\beta(w)) = w$
(2′) $w \vee x = \beta(\beta(w) \vee \beta(x))$
(3′) $\beta(\neg w) = \neg(\neg w)$.

The equivalence of (3′) with (3) is clear. Writing $\beta(x)$ for $w$ in (3′), the result is $\beta(!x) = \neg(!x)$. Conversely, writing $\beta(w)$ for $x$ in (3), the result is $\beta(\neg w) = \neg(\neg w)$.

Consider the 3-valued case, in any $P$-algebra of order 3 using $[0, 1/2, 1]$, the required identities are

$\gamma(x,w) = x \Rightarrow w \vee w \vee x(\neg(!w)) = x \vee x\neg(!x)$, so
$\beta(x) = \gamma(x,0) = \neg x \vee x(\neg(!x))$ with the alternate
$\gamma(x,w) = \beta(x) \vee x \Rightarrow w \vee w = \beta(x) \vee x \rightarrow w$.

Consider finally the 4-valued case, in any $P$-algebra of order 4 using $[0, 1/3, 2/3, 1]$ with the above conditions for involution $\beta(x)$. The required identities are

$$\begin{aligned} \gamma(x,w) = &\ \beta(x) \vee x \Rightarrow w \vee w \vee (\neg(\neg w))x(\neg(!x)) \\ = &\ \beta(x) \vee x \rightarrow w \vee (\neg(\neg w))x(\neg(!x)). \end{aligned}$$

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*Indiana University, Bloomington*
*U.S.A.*