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INTERMEDIATE LOGICS WITHOUT THE INTERPOLATION PROPERTY

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An intermediate logic L has the Interpolation Property (IP) iff for every $\alpha \to \beta \in L$, if $Var(\alpha) \cap Var(\beta) \neq \emptyset$, then there exists a formula γ built up from variables occurring both in α and in β such that $\alpha \to \gamma \in L$, $\gamma \to \beta \in L$ ($Var(\alpha)$ denotes the set of all variables occurring in α).

The following simple lemma will be used in the sequel:

Lemma. *If a formula γ is built up from the variable x only, then $\gamma \to (\neg\neg x \to x) \in INT$ or $\neg\neg x \to \alpha \in INT$.*

The proof reduces to the easy observation of the Rieger-Nishimura algebra.

Theorem 1. *Suppose that L_1, L_2 are intermediate logics such that $L_1 \neq L_1 \cap L_2 \neq L_2$. Then $L_1 \cap L_2$ does not possess the IP. In other words, Hallden-incomplete intermediate logics does not possess the IP.*

Proof. Let α, β be formulas with no common variables such that $\alpha \in L_1 - L_2$, $\beta \in L_2 - L_1$ and let x be a new variable. It is easy to see that $(\alpha \to x) \to (x \vee \beta) \in L_1 \cap L_2$. I shall prove that no formula γ built up from the variable x only is such that $(\alpha \to x) \to \gamma$, $\gamma \to (x \vee \beta) \in L_1 \cap L_2$. It suffices to show (see Lemma) that $(\alpha \to x) \to (\neg\neg x \to x) \notin L_1 \cap L_2$ and $\neg\neg x \to (x \vee \beta) \notin L_1 \cap L_2$ Let $\mathcal{A}$, $\mathcal{Z}$ be strongly compact pseudo-Boolean algebras separating α from L_2 and β from L_1, respectively. The formulas $\neg\neg x \to (x \vee \beta)$ and $(\alpha \to x) \to (\neg\neg x \to)$ are refuted by $\mathcal{Z}$ and $\mathcal{A}$, respectively, so they are not in $L_1 \cap L_2$.

Let $\mathcal{R}$ denote the Rieger-Nishimura algebra, the subalgebra of the Lindenbaum algebra of the intuitionistic propositional logic consisting of the equivalence classes of the formulas φ_n, $n = 0, 1, \ldots$ described in [1], built up from the variable p only. Let $\mathcal{R}_i$ be $\mathcal{R}/([\varphi_i])$ and let $E(\mathcal{R}_i)$ denote the intermediate logic determined by $\mathcal{R}_i$.

THEOREM 2. *$E(\mathcal{R}_{2n+9})$ does not possess the IP for $n = 0, 1, 2, \ldots$.*

PROOF. The formula $(\varphi_{2n+8} \to q) \to ((q \to (\neg r \vee \neg\neg r)) \to (\neg r \vee \neg\neg r))$ where q, r are different variables, different from p, is in $E(\mathcal{R}_{2n+9})$, $n = 0, 1, \ldots$. Now the argument similar to that used in TH 1 ends the proof.

Note that the logics mentioned is TH2 are Hallden-complete (see [2]).

References

[1] A. Wroński, J. Zygmunt, *Remarks on the free pseudo-Boolean algebra with one-element free-generating set*, **Reports on Mathematical Logic** 2 (1974), pp. 77–82.

[2] A. Wroński, *Remarks on Hallden completeness of modal and intermediate logics*, this **Bulletin** 5.4 (1976), pp. 126–130.

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