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A NATURAL DEDUCTION RELEVANCE LOGIC

The relevance logic (NDR) presented in this paper is the result of an attempt to find a natural deduction development, in the style of I. M. Copi (**Introduction to Logic**, 4th ed., MacMillan, 1972), for the relevance logic I presented in "A Three-Valued Interpretation for a Relevance Logic" (**The Relevance Logic Newsletter**, Vol. 1, no. 3, 1976).

The propositional variables of NDR are, $p_1, p_2, \ldots$. NRD's well-formed formulas are constructed in the standard way by using propositional variables, parentheses and the connectives, $-, \cdot$ and $\vee$, in order of increasing binding strength. '$P \supset Q$' is by definition '$-(P \cdot -Q)$'. Capital letters with or without subscripts are metalinguistic variables which range over the well-formed formulas. We will use '$\vdash_r$' to present NDR's rules of inference:

1. $P \vdash_r P \vee Q$, where every p_i in Q occurs in P. (Restricted Addition, RA)
2. $P \vdash_r P \cdot (Q \vee -Q)$, where every p_i in Q occurs in P. (Restricted Tautology Conjunction, RTC)
3. $P, Q \vdash_r P \cdot Q$ (Conjunction, Conj.)
4. $P \cdot Q \vdash_r P$ (Simplification, Simp.)
5. $P \vee Q \cdot R \vdash_r P \vee Q$ (Disjunctive Simplification, DS)
6. $P \vee Q \cdot -Q \vdash_r P$ (Contradiction Elimination, CE)
7. If $S \equiv_l T$ in virtue of exactly one of the following statements then $F(S) \vdash F(T)$.
   i) $P \cdot (Q \vee R) \equiv_l P \cdot Q \vee P \cdot R$ (DeMorgan's, DeM)
   ii) $P \cdot (Q \vee R) \equiv_l P \cdot Q \vee P \cdot R$ (Distribution, Dist.)
   $P \vee Q \cdot R \equiv_l (P \vee Q) \cdot (P \vee R)$

iii)	$P \cdot (Q \cdot R) \equiv_l (P \cdot Q) \cdot R$	(Association, Assoc.)
	$P \vee (Q \vee R) \equiv_l (P \vee Q) \vee R$	
iv)	$P \cdot Q \equiv_l Q \cdot P$	(Computation, Com.)
	$P \vee Q \equiv_l Q \vee P$	
v)	$- - P \equiv_l P$	(Double Negation, DN)
vi)	$P \cdot P \equiv_l P$	(Tautology, Taut.)
	$P \vee P \equiv_l P$	

NDR's entailment relation, symbolized by '$\vdash$', is defined as follows: $P_1, \ldots, P_n \vdash C$ if and only if there is a sequence of well-formed formulas $S_1, \ldots, S_m$ such that $S_m = C$ and each S_i $(1 \leqslant i \leqslant m)$ is either a P_i $(1 \leqslant i \leqslant n)$ or follows from preceding S_j by one of the rules of inference.

THEOREM 1. *If $P_1, \ldots, P_n \vdash C$ then $P_1, \ldots, P_n$ classically entails C and every p_i in C occurs in $P_1, \ldots, P_n$.*

PROOF. Every valuation which assigns t to the premises of the rules of inference assigns t to the conclusion. Furthermore, none of the rules of inference introduce into the conclusion propositional variables which do not occur in the premises.

THEOREM 2. (Indirect Proof.) *If $P \cdot -Q \vdash R \cdot -R$ and every p_i in Q occurs in P then $P \vdash Q$.*

PROOF. Let $S_1, \ldots, S_n$ be a sequence of well-formed formulae such that $S_1 = P \cdot -Q$, $S_n = R \cdot -R$ and each S_i $(1 \leqslant i \leqslant n)$ is either $P \cdot -Q$ or follows from S_j or from S_j and S_k $(1 \leqslant j,\ k < n)$. Then construct this sequence of statements:

1.	P		
2.	$P \cdot (Q \vee -Q)$		1, RTC
$a_1 (= 3)$.	$P \cdot Q \vee P \cdot -Q$	$(P \cdot S \vee S_1)$	2, Dist.
	.		
	.		
	.		
a_2.	$P \cdot Q \vee S_2$		
	.		
	.		
	.		
a_n.	$P \cdot Q \vee S_n$		

$a_n + 1.$	$P \cdot Q$	a_n, CE
$a_n + 2.$	$Q \cdot P$	$a_n + 1$, Com.
$a_n + 3.$	Q	$a_n + 2$, Simp.

The steps from, but excluding, $P \cdot Q \vee S_{j-1}$ to, and including, $P \cdot Q \vee S_j$ for $1 < j \leqslant n$ are to be filled in as follows:

i) If $S_j = P \cdot -Q$ then supply the sequence

$a_j - 1.$	$(P \cdot Q \vee P \cdot -Q) \cdot (Q \vee -Q)$	a_1, RTC
$a_j.$	$P \cdot Q \vee P \cdot -Q$	$a_j - 1$, Simp.

Make $a_j - 2 = a_{j-1}$.

ii) If $S_i \vdash S_j$ $(i < j)$ by RA, where $S_j = S_i \vee T$, then supply the sequence

$a_j - 1.$	$(P \cdot Q \vee S_i) \vee T$	a_i, RA
$a_j.$	$P \cdot Q \vee (S_i \vee T)$	$a_j - 1$, Assoc.

Make $a_j - 2 = a_{j-1}$.

iii) If $S_i \vdash S_j$ $(i < j)$ by RTC, where $S_j = S_i \cdot (T \vee -T)$, then supply the sequence

$a_j - 7.$	$(P \cdot Q \vee S_i) \cdot (T \vee -T)$	a_i, RTC
$a_j - 6.$	$(T \vee -T) \cdot (P \cdot Q \vee S_i)$	$a_j - 7$, Com.
$a_j - 5.$	$(T \vee -T) \cdot (P \cdot Q) \vee$ $(T \vee -T) \cdot S_i$	$a_j - 6$, Dist.
$a_j - 4.$	$(T \vee -T) \cdot S_i \vee (T \vee -T) \cdot$ $(P \cdot Q)$	$a_j - 5$, Com.
$a_j - 3.$	$(T \vee -T) \cdot S_i \vee (P \cdot Q) \cdot$ $(T \vee -T)$	$a_j - 4$, Com.
$a_j - 2.$	$(T \vee -T) \cdot S_i \vee (P \cdot Q)$	$a_j - 3$, DS
$a_j - 1.$	$(P \cdot Q) \vee (T \vee -T) \cdot S_i$	$a_j - 2$, Com.
$a_j.$	$(P \cdot Q) \vee S_i \cdot (T \vee -T)$	$a_j - 1$, Com.

Make $a_j - 8 = a_{j-1}$.

iv) If $S_h, S_i \vdash S_j$ $(h, i < j)$ by Conj., where $S_j = S_h \cdot S_i$, then supply the sequence

$a_j - 1.$	$(P \cdot Q \vee S_n) \cdot (P \cdot Q \vee S_i)$	a_h, a_i Conj.
$a_j.$	$P \cdot Q \vee (S_h \cdot S_i)$	$a_j - 1$, Dist.

Make $a_j - 2 = a_{j-1}$.

Procedures for filling in the lines between a_j and a_{j-1} when $S_i \vdash S_j$ in virtue of Rules 4-7 are also easily constructed.

Theorem 3. (Transitivity of Entailment.) *If $P \vdash Q$ and $Q \vdash R$ then $P \vdash R$.*

Proof. Let S_1 $(= P), S_2, \ldots, S_m$ $(= Q)$ be a sequence of well-formed formulas which shows that $P \vdash Q$ and let S_m $(= Q)$, $S_{m+1}, \ldots, S_n$ $(= R)$ be a sequence of well-formed formulas which shows that $P \vdash R$. Then $S_1, \ldots, S_n$ shows that $P \vdash R$.

Theorem 4. *If P classically entails Q and every p_i in Q occurs in P then $P \vdash Q$.*

Proof. Assume the antecedent. Then $P \cdot -Q$ is a contradiction. By DeM, Dist., Assoc., Com., DN and Taut. $P \cdot -Q \vdash R_1 \cdot -R_1 \cdot S_1 \vee \ldots \vee R_n \cdot -R_n \cdot S_n \cdot (R_1 \cdot -R_1 \cdot S_1 \vee \ldots \vee R_n \cdot -R_n \cdot S_n$ is one of the formulas which will be produced when following some of the various mechanical procedures for generating the disjunctive normal form of $P \cdot -Q$). By CE and Simp. $R_1 \cdot -R_1 \cdot S_1 \vee \ldots \vee R_n \cdot -R_n \cdot S_n \vdash R_1 \cdot -R_1$. By Theorem 3 (Th. 3), $P \cdot -Q \vdash R_1 \cdot -R_1$. By Th. 2 $P \vdash Q$.

Theorem 5. (Adjunction). *If $P \vdash Q$ and $P \vdash R$ then $P \vdash Q \cdot R$.*

Proof. Let $S_1, \ldots, S_m$ $(= Q), \ldots, S_n$ $(= R)$, where $m \leqslant n$, be a sequence that shows that $P \vdash Q$ and $P \vdash R$. Let $S_{n+1} = Q \cdot R$. Then $S_1, \ldots, S_{n+1}$ shows that $P \vdash Q \cdot R$, using Conj.

Theorem 6. (Deduction Theorem). *If $P \cdot Q$ and every p_i in Q occurs in P then $P \vdash Q \supset C$.*

Proof. Assume the antecedent. By Theorem 1 $P \cdot Q$ classically entails C. Then P classically entails $Q \supset C$. Since every p_i in Q occurs in P and every p_i in C occurs in $P \cdot Q$ it follows that every p_i in $Q \supset C$ occurs in P. By Theorem 4 $P \vdash Q \supset C$.[1]

Theorem 7. (Antilogism). *If $P \cdot Q \vdash R$ and every p_i in Q occurs in P then $P \cdot -R \vdash -Q$.*

Proof. By Simp. $P \cdot -R \vdash P$. Assume the antecedent. By Th. 6 and the definition of '$\supset$' $P \vdash -(Q \cdot -R)$. By Th. 3 $P \cdot -R \vdash -(Q \cdot -R)$. By Com.

[1]This proof, suggested by Richard Routley, is more straightforward than my original proof. I am grateful for Professor Routley's comments, which led to several improvements.

and Simp. $P \cdot -R \vdash -R$. By Th. 5 $P \cdot -R \vdash -R \cdot -(Q \cdot -R)$. By Dem, Dist., Com. and Simp. $-R \cdot -(Q \cdot -R) \vdash -Q$. By Th. 3 $P \cdot -R \vdash -Q$.

The difference between NDR and the relevance logic presented in "A Three-Valued Interpretation of a Relevance Logic" is that the latter does not recognize the validity of any arguments with contradictory premises, whereas NDR does. For example, $p_1 \cdot -p_1 \vdash p_1$ in NDR. But both of these logics endorse what W. T. Parry (The Logic of C. I. Lewis', **The Philosophy of C. I. Lewis**, ed. P. A. Schilpp, 1968, pp. 115–54) called the Proscriptive Principle, which keeps those arguments which contain a p_i that occurs in the conclusion but not in a premise from being valid. Charles Kielkopf ('Adjunction and Paradoxical Derivations', **Analysis**, Vol. 35, no. 4, 1975, pp. 127–9) showed that the system which Parry based on the Proscriptive Principle inadvertently permits the derivation of any statement from a contradiction.

Perhaps the most worrisome feature of NDR is that it denies that in general if A entails B then $-B$ entails $-A$. For example, though $p_1 \cdot p_2$ entails p_1 it is false that $-p_1$ entails $-(p_1 \cdot p_2)$. But the reservations which beginning students of logic have about the validity of Unrestricted Addition, which would guarantee that $-p_1$ entails $-p_1 \vee -p_2$ suggest that this apparent defect may be a virtue.[2]

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