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BARCAN FORMULAS IN *SCI* WITH QUATIFIERS

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In this paper some syntactical properties of theories in a propositional language L containing the identity connective and propositional quantifiers are considered. The theories under consideration are based on a non-Fregean logic as described in [1], [2].

§1. The language L has the following logical primitive symbols:

a) propositional variables: $p, q, r \dots$
b) connectives: $\neg$ (negation), $\wedge$ (conjunction), $\rightarrow$ (implication), $\leftrightarrow$ (equivalence), and $\equiv$ (identity)
c) quantifiers: $\forall$ (universal) and $\exists$ (existential)
d) some special connectives: the constants $0, 1$ and the theory connectives $\Diamond$ and $\Box$.

By Cn we denote the consequence operation or deducibility relation over L which is characterized as in [1] by some axioms and rules. It should be noted that no axiom involves explicitly connectives listed in d), and the universal closure of

(1) $\forall c(\alpha \equiv \beta) \rightarrow (Qv\alpha \equiv Qv\beta)$, where Q stands for $\forall$ or $\exists$ is among the axioms.

Cn has the following property [2]:

$\ulcorner \alpha \equiv \beta \urcorner \in Cn(\emptyset)$ iff α and β are congruent, i.e. one of them can be obtained from the other by changing some bound variables.

Let T_0 be a theory of the logic Cn in L based on the axioms:

(a_1) $0 \equiv \forall_p p$
(a_2) $1 \equiv \exists_p p$
(a_3) $\forall_p(\Box p \equiv (p \equiv 1))$
(a_4) $\forall_p(\Diamond p \equiv \neg(p \equiv 0))$

We ask the question: What are necessary and sufficient conditions for universal closures of the schemas:

$(B_1 \rightarrow)$ $\forall v \Box \alpha \rightarrow \Box \forall v$
$(B_2 -)$ $\Diamond \exists v \alpha \rightarrow \exists v \Diamond \alpha$

to be theorems of any theory T of Cn such that $T \supseteq T_0$?

Theorem 1. *Formulas* $(B_1 \rightarrow)$ *and* $(B_2 \rightarrow)$ *are theorems of* T *iff* (i_1) *and* (i_2) *are theorems of* T*, where*

(i_1) $\forall v 1 \equiv 1$
(i_2) $\exists v 0 \equiv 0.$

Proof. Let us assume that $(B_1 \rightarrow)$ is a theorem of T. Replacing by 1 in $(B_1 \rightarrow)$ we have:

(2) $\forall v(1 \equiv 1) \rightarrow (\forall v 1 \equiv 1)$

Antecendent of (2) is a theorem of our logic, hence (i_1) is a theorem of T. The proof of (i_2) is analogous.
Now, form (1) we have

(3) $\forall v(\alpha \equiv 1) \rightarrow (\forall v \alpha \equiv \forall v 1)$

By (i_1) and (a_3) this gives $(B_1 \rightarrow)$.
Similarly, one can easily deduce $(B_2 \rightarrow)$ from (1), (i_2), (a_4) and de Morgan laws.

§2. Sufficient conditions for equivalential and equational Barcan formulas.

Let L_1 be a language obtained by adding to L a new binary connective $\leqslant$. We restrict our consideration to those theories T in L_1 which satisfy the following conditions:

(w_1) $T_0 \subset T$
(w_2) T is an invariant theory, i.e. it is closed under the rule of universal generalization
(w_3) The formulas $(p_1), (p_2), (p_3)$ listed below are theorem of T:
- (p_1) $p \leqslant p$
- (p_2) $(p \leqslant q) \wedge (q \leqslant p) \rightarrow (p \equiv q)$
- (p_3) $((p \leqslant q) \wedge (q \leqslant r)) \rightarrow (p \leqslant r)$

THEOREM 2. *If the following formulas*

(s_1) $\forall v\alpha \leqslant \alpha[^v/_\beta]$
(s_2) $\forall v(w \leqslant \alpha) \rightarrow (w \leqslant \forall v\alpha)$
(s_3) $\alpha \leqslant \exists v\alpha$
(s_4) $\forall v(\alpha \leqslant w) \rightarrow (\exists v\alpha \leqslant w)$

are theorems of T, then the formulas

($B_1 \leftrightarrow$) $\forall v\Box\alpha \leftrightarrow \Box\forall v\alpha$
($B_2 \leftrightarrow$) $\Diamond\exists v\alpha \leftrightarrow \exists v\Diamond\alpha$

are also theorems of T.

PROOF. Let us assume (s_1) and (s_2). That can be expressed equivalently by the formula

(Q_1) $(v_0 \equiv \forall v\alpha) \leftrightarrow (\forall v(v_0 \leqslant \alpha) \wedge \forall w(\forall v(w \leqslant \alpha) \rightarrow (w \leqslant v_0)))$

Analogously, (s_3) and (s_5) are equivalent to:

(Q_2) $(v_0 \equiv \exists v\alpha) \leftrightarrow (\forall v(\alpha \leqslant v_0) \wedge \forall w(\forall v(\alpha \leqslant w) \rightarrow (v_0 \leqslant w)))$

Replacing in (Q_1) v_0 and α by 1 we have (i_1), which is equivalent to ($B_1 \rightarrow$). For the converse implication, let us observe that (s_1), (s_3), (p_3) and (Q_2) imply:

(4) $\forall v(\alpha \leqslant 1)$

Replacing in (Q_1) r by 1 we have

(5) $(1 \equiv \forall v\alpha) \rightarrow \forall v(1 \leqslant \alpha)$

From (4) and (5) it follows that

(6) $(1 \equiv \forall v\alpha) \rightarrow \forall v(\alpha \leqslant 1 \wedge 1 \leqslant \alpha)$

This by (p_2) gives $(B_1 \leftrightarrow)$. The proof of $(B_2 \leftrightarrow)$ is similar. To give sufficient conditions for the equational Barcan formulas

$(B_1 \equiv)$ $\forall v\Box\alpha \equiv \Box\forall v\alpha$
$(B_2 \equiv)$ $\Diamond\exists v\alpha \equiv \exists v\Diamond\alpha$

to hold true, we confine ourselves to theories T in L_1 which satisfy conditions (w_1), (w_2), (w_3), and $(s_1), \ldots, (s_4)$ together

(s_5) $\forall v(\alpha \equiv \beta) \leqslant (Qv\alpha \equiv Q\beta)$ where Q is $\forall$ or $\exists$

are schemas of theorems of T.
It should be noted that (s_5) is a strengthening of (1)

THEOREM 3. *If the formula*

(f_1) $(p \leqslant q) \rightarrow (\Box p \leqslant \Box q)$

or the formulas

(f_2) $(p \leqslant q) \rightarrow (\neg q \leqslant \neg p)$
(f_3) $(p \leqslant q) \rightarrow (q \equiv 0 \leqslant p \equiv 0)$
(f_4) $\neg\forall v\alpha \equiv \exists v\neg\alpha$

are theorems of T*, then* $(B_1 \equiv)$ *are also schemas of theorems of* T*.*

PROOF. From (s_5) and (i_1) it follows

(7) $\forall v(\alpha \equiv 1) \leqslant (\forall v\alpha \equiv 1)$

To deduce the converse inequality we take into account (f_1), (s_2), (s_1):

(8) $(\forall v\alpha \leqslant \alpha) \rightarrow (\Box\forall v\alpha \leqslant \Box\alpha)$
(9) $(\Box\forall v\alpha \leqslant \Box\alpha) \rightarrow (\Box\forall v\alpha \leqslant \forall v\Box\alpha)$

By aplication to (8) and (9) of modus ponens abd by (7), (a_3) and (p_2) we have $(B_1 \equiv)$. In a similar way from (f_2), (f_3) and (f_4) we deduce $(B_2 \equiv)$.

§3. Barcan formulas in modal logics versus Barcan formulas in SCI

Schemas ($B_1 \to$) and ($B_2 \to$) are called Barcan formulas and they are theorems of the modal predicate calculus $S5$ and the Brouwerian system. They are mutually equivalent on the ground of the system T of Feys but they are theses neither of T nor S_4. However, the converse implications

($B_1 \leftarrow$) $\Box\forall v\alpha \to \forall v\Box\alpha$
($B_2 \leftarrow$) $\exists v\Diamond\alpha \to \Diamond\exists v\alpha$

are theorems of T. It follows that in $S5$ and in the Brouwerian system the equivalential and strict equivalential Barcan formulas hold to be true. Whereas in SCI with quantifiers the Barcan formulas hold to be true on lowest levels and ($B_1 \to$) and ($B_2 \to$) are not equivalent. In the present paper it has been shown that the converse implications to the Barcan formulas are consequences of some assumptions about the connective $\leqslant$.

References

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