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A METHOD OF AXIOMATIZING AN INTERSECTION OF PROPOSITIONAL LOGICS

This is a summary of a lecture read at the Seminar of the Section of Logic, Polish Academy of Sciences, held by Professor Ryszard Wójcicki in Wrocław, February 1977.

By a language we shall mean in this paper a propositional language with an infinite of variables and an arbitrary set of connectives including the binary connective $\rightarrow$, the implication. To make notations more readable we adopt the convention of associating to the left and ignoring the implication sign $\rightarrow$. For example we will write $x(xy)(xy)$ instead of $(x \rightarrow (x \rightarrow y)) \rightarrow (x \rightarrow y)$ and $xyy(yxx)$ instead of $((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x)$. By a system in a given language we mean a subset of this language which is closed under the substitution rule of this language and the detachment rule for the implication. A subset of a system is called a basis iff it is not contained in any proper subsystem of this system. A system is finitely axiomatizable iff it has a finite basis.

In this paper we give a sufficient condition of finite axiomatizability of the intersection of two finitely axiomatizable systems. A trick used in the proof of sufficiency of this condition can be applied for showing that the intersection of the implicational fragments of the intuitionistic propositional logic and the infinite-valued logic of Łukasiewicz can be axiomatized by the following axioms:

(l_1) $x(yz)$,
(l_2) $xy(yz(xz))$,
(l_3) $xy(yz)(yx)$,
(l_4) $x(xy)(xy)s(zuu(uzz)ss)$.

Let $D = \{\Pi_1, \ldots, \Pi_4\}$ where $\Pi_1, \ldots, \Pi_4$ are the following purely implicational formulas:

$\Pi_1 = x(yx)$,
$\Pi_2 = x(xyy)$,
$\Pi_3 = xy(yz(xz))$,
$\Pi_4 = xyy(yz(yz(xzz))(yz(xzz)))$.

If α, β are formulas then for every natural number $n \in N = \{1, 2, \ldots\}$ we define the abbreviation $\alpha^n\beta$ putting $\alpha^1\beta = \alpha\beta$ and $\alpha^{n+1}\beta = \alpha(\alpha^n\beta)$. We shall make frequent use of the following:

Lemma. *If* $\mathbf{T}$ *is a system containing* D *and* $\alpha, \beta, \gamma, \delta$ *are formulas of the language of* $\mathbf{T}$ *then the following formulas belong to* $\mathbf{T}$*:*

(i) $\alpha(\beta\gamma)(\beta(\alpha\gamma))$,
(ii) $\alpha\alpha$,
(iii) $\alpha\beta\gamma\gamma(\alpha\gamma\gamma(\beta\gamma\gamma))$,
(iv) $\alpha\beta(\gamma^n\alpha(\gamma^n\beta))$,
(v) $\alpha^n(\beta\gamma)(\beta(\alpha^n\gamma))$,
(vi) $\beta(\alpha^n\gamma)\alpha^n(\beta\gamma))$,
(vii) $\alpha\beta\beta((\beta\gamma)^{n+1}(\alpha\gamma\gamma)((\beta\gamma)^n(\alpha\gamma\gamma)))$,
(viii) $\delta^n(\alpha\beta\gamma\gamma)(\delta^m(\alpha\gamma\gamma)(\delta^{n+m}(\beta\gamma\gamma)))$.

Theorem 1. *If* $\mathbf{T}_0, \mathbf{T}_1$ *are finitely axiomatizable systems in the same language and* $D \subseteq \mathbf{T}_0 \cap \mathbf{T}_1$ *the* $\mathbf{T}_0 \cap \mathbf{T}_1$ *is a finitely axiomatizable system in this language.*

Sketch of Proof. Let us suppose that $\mathbf{T}_0, \mathbf{T}_1$ are finitely axiomatizable systems in the language $\mathbf{L}$ and $D \subseteq \mathbf{T}_0 \cap \mathbf{T}_1$. Let $T_0 = \{\chi_0, \ldots, \chi_{k_0}\}$ be a basis of $\mathbf{T}_0$ and $T_1 = \{\xi_0, \ldots, \xi_{k_1}\}$ a basis of $\mathbf{T}_1$. Since the language $\mathbf{L}$ has an infinite number of variables then we can assume that $Var(T_0) \cap Var(T_1) = \emptyset$ and there exists a variable x such that $x \notin Var(T_0 \cup T_1)$. We claim that the set $T = \{\chi_i x(\xi_j xx) : i \leqslant k_0, j \leqslant k_1\} \cup D$ is a basis of $\mathbf{T}_0 \cap \mathbf{T}_1$. Let $\mathbf{T}$ be the smallest system in the language $\mathbf{L}$ containing T. Applying Π_1, Π_2 and the condition (i) of Lemma it is easy to prove that $\mathbf{T} \subseteq \mathbf{T}_0 \cap \mathbf{T}_1$. In order to prove the converse inclusion let us note first the following:

(1) $\forall_{\alpha \in \mathbf{T}_1} \forall_{\chi \in T_0} \forall_{\gamma \in \mathbf{L}} \exists_{n \in N} (\chi\gamma)^n(\alpha\gamma\gamma) \in \mathbf{T}$.

Put $\mathbf{C} = \{\alpha \in \mathbf{L} : \forall_{\chi \in T_0} \forall_{\gamma \in \mathbf{L}} \exists_{n \in N} (\chi\gamma)^n(\alpha\gamma\gamma) \in \mathbf{T}$, then it is easy to see that every substitution of an axiom of $\mathbf{T}_1$ belongs to $\mathbf{C}$ and therefore we need only to prove that $\mathbf{C}$ is closed under the detachment rule. Suppose that $\alpha, \alpha\beta \in C$, then for every $\chi \in T_0$ and for every $\gamma \in \mathbf{L}$ there exist $m, n \in N$ such that $(\chi\gamma)^m(\alpha\gamma\gamma)$, $(\chi\gamma)^n(\alpha\beta\gamma\gamma) \in \mathbf{T}$. By the condition (viii) of Lemma we get that $(\chi\gamma)^{n+m}(\beta\gamma\gamma) \in \mathbf{T}$ which proves that $\beta \in \mathbf{C}$. Next one obtains the following strengthened version of (1):

$(1')$ $\quad \forall_{\alpha \in \mathbf{T}_1} \forall_{\chi \in \mathbf{T}_0} \forall_{\gamma \in \mathbf{L}} \chi\gamma(\alpha\gamma\gamma) \in \mathbf{T}.$

Indeed, let us suppose that $\alpha \in \mathbf{T}_1$, $\chi \in \mathbf{T}_0$ and $\gamma \in \mathbf{L}$, then by (1) we get that for some $n \in N$, $(\chi\chi)^n(\alpha\chi\chi) \in \mathbf{T}$ and consequently $\alpha\chi\chi \in \mathbf{T}$ by virtue of the condition (ii) of Lemma. Applying now the condition (vii) of Lemma we get that for every $i \in N$, $(\chi\gamma)^{i+1}(\alpha\gamma\gamma)(\chi\gamma)^i(\alpha\gamma\gamma)) \in \mathbf{T}$ and therefore the condition $(1')$ follows directly from (1).

Next, using a similar argument one can prove successively:

(2) $\quad \forall_{\alpha \in \mathbf{T}_1} \forall_{\beta \in \mathbf{T}_0} \forall_{\gamma \in \mathbf{L}} \exists_{n \in N} (\beta\gamma)^n(\alpha\gamma\gamma) \in \mathbf{T}$

$(2')$ $\quad \forall_{\alpha \in \mathbf{T}_1} \forall_{\beta \in \mathbf{T}_0} \forall_{\gamma \in \mathbf{L}} \beta\gamma(\alpha\gamma\gamma) \in \mathbf{T}.$

Finally, supposing that $\delta \in \mathbf{T}_0 \cap \mathbf{T}_1$ we get by $(2')$ that $\delta\delta(\delta\delta\delta) \in \mathbf{T}$ and consequently $\delta \in \mathbf{T}$ by virtue of the condition (ii) of Lemma. This proves that $\mathbf{T} = \mathbf{T}_0 \cap \mathbf{T}_1$ Q.E.D.

Now let $\mathbf{F}$ be the language with the ordinary connectives $\to, \wedge, \vee, 0$ (implication, conjunction, disjunction, falsum). Let $\mathbf{I}$ and $\mathbf{L}$ be the system in $\mathbf{F}$ consisting of all theorems of the intuitionistic propositional logic and all theorems of the infinite-valued logic of Łukasiewicz respectively. For every set of connectives $\triangle \subseteq \{\to, \wedge, \vee, 0\}$ let $\mathbf{F}^\triangle$ be the sublanguage of $\mathbf{F}$ determined by $\triangle$ and let $\mathbf{I}^\triangle = \mathbf{I} \cap \mathbf{F}^\triangle$, $\mathbf{L}^\triangle = \mathbf{L} \cap \mathbf{F}^\triangle$. Let $B_{\{\to\}} = \{\iota_1, \ldots, \iota_4\}$, $B_{\{\wedge\}} = \{\kappa_1, \kappa_2, \kappa_3\}$, $B_{\{\vee\}} = \{\delta_1, \delta_2, \delta_3\}$ and $B_{\{0\}} = \{\varphi\}$ where:

$\iota_1 = x(yx)$,
$\iota_2 = xy(yz(xz))$,
$\iota_3 = xy(yx)(yx)$,
$\iota_4 = x(xy)(xy)s(zuu(uzz)ss)$,
$\kappa_1 = (x \wedge y)x$,
$\kappa_2 = (x \wedge y)y$,
$\kappa_3 = xy(xz(x(y \wedge z)))$,

$\delta_1 = x(x \vee y)$,
$\delta_2 = y(x \vee y)$,
$\delta_3 = xy(zy((x \vee z)y))$,
$\varphi = Qx$.

Observing that $D \subseteq \mathbf{I}^{\{\to\}} \cap \mathbf{L}^{\{\to\}}$ and applying the trick used in the proof of Theorem 1 one gets the following:

THEOREM 2. *If* $\to \in \triangle \subseteq \{\to, \wedge, \vee, 0\}$ *then* $\bigcup(B_{\{c\}} : c \in \triangle)$ *is a basis of the system* $\mathbf{I}^{\triangle} \cap \mathbf{L}^{\triangle}$.

REMARK. Let $\mathbf{D}$ be the smallest purely implicational system containing D, then $\mathbf{D} \subsetneq \mathbf{I}^{\triangle} \cap \mathbf{L}^{\triangle}$ since $\iota_4 \notin \mathbf{D}$ as it can be shown by means of the matrix $\langle\{0,1,2,3\}, \to, \{0\}\rangle$ such that $0 \to 1 = 2 \to 3 = 1$, $0 \to 2 = 1 \to 2 = 1 \to 3 = 2$, $0 \to 3 = 3$ and $a \to b = 0$ otherwise.

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