Bulletin of the **S**ection of **L**ogic
Volume 6/4 (1977), pp. 186–188

reedition 2011 [original edition, pp. 186–189]

Barbara Woźniakowska

ALGEBRAIC PROOF OF THE SEPARATION THEOREM FOR THE INFINITE-VALUED LOGIC OF ŁUKASIEWICZ

This is an abstract of the paper submitted to Reports on Mathematical Logic.

By D we shall always mean a set of connectives included in $\{\rightarrow, \wedge, \vee, \neg\}$ containing the implication connective $\rightarrow$. We introduce some notations using D as a parameter and adopt the convention to omit D in the case $D = \{\rightarrow, \wedge, \vee, \neg\}$. By D-formula we mean a formula built up in the usual way be means of variables from an infinite set V and the connectives from D. The symbol F_D denotes the set of all D-formulas. The symbol Ł denotes the set of all formulas from F which are provable in infinite-valued logic of Łukasiewicz (see J. Łukasiewicz [4]). An axiomatization of Ł can be obtained by adopting the detachment rule, the substitution rule and the following set of axioms:

$(a0)$ $x \rightarrow (y \rightarrow x)$
$(a1)$ $((x \rightarrow y) \rightarrow (x \rightarrow z)) \rightarrow ((y \rightarrow x) \rightarrow (y \rightarrow z))$
$(a2)$ $((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x)$
$(a3)$ $(x \wedge y) \rightarrow x$
$(a4)$ $(x \wedge y) \rightarrow y$
$(a5)$ $(x \rightarrow y) \rightarrow ((x \rightarrow z) \rightarrow (x \rightarrow (y \wedge z)))$
$(a6)$ $x \rightarrow (x \vee y)$
$(a7)$ $y \rightarrow (x \vee y)$
$(a8)$ $(x \rightarrow y) \rightarrow ((z \rightarrow y) \rightarrow ((x \vee z) \rightarrow y)$
$(a9)$ $(\neg x \rightarrow \neg y) \rightarrow (y \rightarrow x)$

We say that a formula is D-provable iff it can be proved from the axioms above by means of a proof being a sequence of D-formulas. The symbol $Ł_D$ denotes the set of all D-provable formulas.

By D-identity we mean an expression of the form $\alpha = \beta$ where α, β are D-formulas. By D-algebra we mean an algebra of type determined by the connectives from D satisfying all D-identities from the following list:

$(i0)$ $(x \to (y \to x)) \to z = z$
$(i1)$ $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$
$(i2)$ $(x \to y) \to y = (y \to x) \to x$
$(i3)$ $(x \wedge y) \to x = x \to x$
$(i4)$ $(x \wedge y) \to y = x \to x$
$(i5)$ $(x \to z) \to (x \to (y \wedge z)) = (x \to z) \to (x \to y)$
$(i6)$ $x \to (x \vee y) = x \to x$
$(i7)$ $y \to (x \vee y) = x \to x$
$(i8)$ $(z \to y) \to ((x \vee z) \to y) = (z \to y) \to (x \to y)$
$(i9)$ $\neg x \to \neg y = y \to x$

Let K_D be the variety of all D-algebras, then we have the following:

COMPLETENESS LEMMA. $Ł_D = \{\alpha : \alpha = \alpha \to \alpha \in Id(K_D)\}$.

Following A. Tarski [7] we say that the varieties K_0, K_1 (possibly of different types) are polynomially equivalent iff there exists a bijection $\varphi : K_0 \to K_1$ such that for every $\mathcal{A} \in K_0$ the algebras $\mathcal{A}$ and $\varphi(\mathcal{A})$ have the same set of polynomials.

LEMMA.

(i) *The varieties* $K_{\{\to\}}$, $K_{\{\to,\vee\}}$ *are polynomially equivalent;*
(ii) *The varieties* $K_{\{\to,\wedge\}}$, $K_{\{\to,\wedge,\vee\}}$ *are polynomially equivalent;*
(iii) *The varieties* $K_{\{\to,\neg\}}$, $K_{\{\to,\wedge,\neg\}}$, $K_{\{\to,\vee,\neg\}}$, $K_{\{\to,\vee,\wedge,\neg\}}$ *are polynomially equivalent.*

EMBEDDING LEMMA. *If* $D_0 \subseteq D_1$ *then every algebra from* K_{D_0} *can be embedded into* D_0*-reduct of some algebra from* K_{D_1}.

From Completeness Lemma and Embedding Lemma we obtain

SEPARATION THEOREM. $Ł_D = Ł \cap F_D$.

Remark. In the case $D = \{\rightarrow\}$ the Separation Theorem was proved by A. Rose [6] and R. K. Meyer [5]. The varieties for $Ł_{\{\rightarrow\}}$ and Ł were characterized by B. Bosbach [1] and C. C. Chang [2] respectively. The algebraic technique used in this paper is similar to that of A. Horn [3].

References

[1] B. Bosbach, *Residuation groupoids*, **Bulletin de l'Academie Polonaise des Sciences**, vol. XXII, No. 2 (1974), pp. 103–104.

[2] C. C. Chang, *Algebraic analysis of many valued logics*, **Trans. Amer. Math. Soc.**, vol. 88 (1958), pp. 467–490.

[3] A. Horn

[4] J. Łukasiewicz, A. Tarski, *Untersuchungen über den Aussagenkalkul*, **Comptes rendus de la Societe des Sciences et des Lettres de Varsovie Cl.III**, 23 (1930), pp. 1–21.

[5] R. K. Meyer, *Pure denumerable Łukasiewicz implication*, **The Journal of Symbolic Logic** 31 (1966), pp. 575–580.

[6] A. Rose, *Formalization du calcul propositionel implicatif á* $\aleph_0$*-valeurs de Łukasiewicz*, **Comptes rendus hebdomadaires des seances de l'Academie des Sciences**, vol. 243 (1956), pp. 1183–1185.

[7] L. Henkin, J. D. Monk, A. Tarski, **Cylindric Algebras**, North-Holland Publishing Company, Amsterdam-London 1971, 125.

Department of Logic
Jagiellonian University