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A REMARK ON MAXIMAL MATRIX CONSEQUENCES

This is an abstract of a lecture read at the Seminar of the Section of Logic, Polish Academy of Sciences, Wrocław, April 1977.

Let $\underline{A} = (A, f_1, \ldots, f_n)$ be an algebra similar to some propositional language $\underline{S} = (S, F_1, \ldots, F_n)$ free generated by some set $V = \{p_1, p_2, \ldots\}$ of propositional variables. When $a \in A$, $[a]$ denotes the subalgebra of $\underline{A}$ generated by a. A *matrix* is any pair $(\underline{A}, B)$, $B \subseteq A$ (B being the set of designated elements). If $\underline{M}$ is a matrix, then $C_{\underline{M}}$ denotes the *matrix consequence* (cf. [1]) given by $\underline{M}$ in $\underline{S}$. Let C be any consequence operation in $\underline{S}$. C is said to be *structural* provided that $eC(X) \subseteq C(eX)$ for any $X \subseteq S$ and for any endomorphism e of $\underline{S}$. C is said to be *maximal* (*almost-maximal*) if for any structural C_1, $C < C_1$ implies $C_1(\emptyset) = S$ ($C_1(\alpha) = S$ for every $\alpha \in S$). Clearly

(I) C is almost-maximal iff for every structural C_1 such that $C < C_1$, there is some $p \in V$ such that $C_1(p) = S$.

(II) If C is maximal, then C is almost-maximal.

Theorem. *Let for some $a \in A$, $[a] = \underline{A}$ and let $\underline{M} = (\underline{A}, \{a\})$. Then $C_{\underline{M}}$ is almost-maximal.*

Proof. Suppose that $C_{\underline{M}} < C$ for some structural C. Then there are $\alpha \in S$, $X \subseteq S$ such that

(A) $\alpha \in C(X)$ and $\alpha \notin C_{\underline{M}}(X)$.

Hence there is some valuation (homomorphism) h of $\underline{S}$ in $\underline{A}$ such that

(B) $hX \subseteq \{a\}$ and $h\alpha \neq a$.

Let $\beta_1(p)$, $\beta_2(p)$, $\beta_3(p), \ldots$ be any formulas built up of only one variable $p \in V$ such that $\beta_1(a) = hp_1$, $\beta_2(a) = hp_2$, $\beta_3(a) = hp_3, \ldots$. Since $[a] = \underline{A}$ the β's do exist. Define a substitution (endomorphism) e_0 by putting

(C) $e_0 p_i = \beta_i(p)$, all $i \in \omega$.

Clearly, if h_0 is any valuation such that $h_0 p = a$, then $h_0 e_0 X = hX \subseteq \{a\}$ by (B). Then

(D) $e_0 X \subseteq C_{\underline{M}}(p) \subseteq C(p)$.

By structurality of C, (A) implies $e_0\alpha \in C(e_0 X)$; so by (D):

(E) $e_0\alpha \in C(p)$.

If p takes the designated value, then $e_0\alpha$ does not (by (B), (C)). Hence $\{e_0\alpha, p\}$ is not satisfiable in $\underline{M}$. Thus

(F) $C_{\underline{M}}(e_0\alpha, p) = S$.

From (E), (F) we obtain $S = C_{\underline{M}}(e_0\alpha, p) \subseteq C(e_0\alpha, p) = C(p)$, which, on the grounds of (I), concludes the proof.

Corollary 1 (Wójcicki-Wroński). *Let $\underline{A}$ have no proper subalgebras, $a \in A$, $\underline{M} = (\underline{A}, \{a\})$. Then $C_{\underline{M}}$ is almost-maximal.*

Corollary 2. *Let $\underline{A}$, $\underline{M}$ be as in Corollary 1. If $C_{\underline{M}}(\emptyset) \neq \emptyset$, then $C_{\underline{M}}$ is maximal.*

Corollary 3. *Let $B \subseteq A$, $\underline{M} = (\underline{A}, B)$. If every constant function is definable in $\underline{A}$, then $C_{\underline{M}}$ is maximal.*

References

[1] J. Łoś and R. Suszko, *Remarks on sentential logics*, **Indagationes Mathematicae** 20 (1958), pp. 177–183.

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