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Richard Routley

LEWIS'S CALCULUS OF ORDINARY INFERENCE (AS AMENDED 1920 AND 1977)

In his original presentation of The System of Strict Implication ([1], chapter V) Lewis considers a 'partial system contained in Strict Implication' to which he believes some interest attaches.

> If our aim be to create a workable calculus of deductive inference, we shall need to retain the relation of logical product, p & q, but material implication, $p \supset q$, and probably also material sum, $p \vee q$, may be rejected as not sufficiently useful to be worth complicating the system with. The ideas of possibility and impossibility also are unnecessary complications. Such a system may be called the Calculus of Ordinary Inference ([1], p. 318; with modern notation for connectives).

The undefined connectives of the Calculus of Ordinary Inference (COI, for short) are $\rightarrow$, &, $\sim$. Further connectives are defined by Lewis thus: $A \wedge B =_{Df} \sim A \rightarrow B$; $A \circ B =_{Df} \sim (A \rightarrow \sim B)$; $A \leftrightarrow B =_{Df} (A \rightarrow B)$ & $(B \rightarrow A)$; $A \vee B =_{Df} \sim (\sim A$ & $\sim B)$. The axioms of COI are as follows (with Lewis's labels prefixed):

$E.$ $\sim p \rightarrow q \rightarrow . \sim q \rightarrow p$	$F.$ $p \,\&\, q \rightarrow p$
$G.$ $p \rightarrow p \,\&\, p$	$H.$ $p \,\&\, (q \,\&\, r) \rightarrow q \,\&\, (p \,\&\, r)$
$I.$ $p \rightarrow \sim\sim p$	$J.$ $(p \rightarrow q) \,\&\, (q \rightarrow r) \rightarrow .p \rightarrow r$
$K.$ $p \,\&\, q \rightarrow p \circ q$	$L.$ $p \,\&\, q \rightarrow r \,\&\, s \rightarrow .p \circ q \rightarrow r \circ s$

Lewis gives no rules of inference for COI, yet clearly expects theorems to emerge. He appears to have assumed (in [1] and [2]) that the rules are the same as those of the systems of strict implication, i.e. of the system (subsequently called) $S3$. Let us take it then that the rules are the same as those of $S3$; that is, the rules of COI are Substitution (upon sentential variables), Modus Ponens (i.e. $A, A \to B \to B$), Adjunction (i.e. $A, B \to A \;\&\; B$) and Replacement (i.e. $A \leftrightarrow B, C(A) \to C(B)$).

Unfortunately COI turns out, as we shall see, to be nothing but classical two-valued logic, differently formulated and with $\to$ serving for material implication – hardly a logic of ordinary inference. By 1920 however, Lewis had already realised that axiom L of COI was false; and he proposed in [2] replacing it by one hals of L, namely

$$L'. \quad (p \;\&\; q \to r \;\&\; s) \to .p \circ q \to r \circ s.$$

Call the amended system, with L' replacing L, $COIA$. But this – in other ways most desirable – weakening of L apparently blocks the derivation of &-Commutation, $p \;\&\; q \to .q \;\&\; p$, which was obtainable from L, essentially as follows:

1. $p \circ q \to .q \circ p$, by E, I, J, Replacement, and definition of $\circ$.
2. $p \;\&\; q \to q \;\&\; p$, by I and L, with Modus Ponens.

Conjecture 1: &-Commutation is not a theorem of $COIA$.

Remark: It was initially thought that matrix methods would quickly establish the result. Not so. Furthermore both R. K. Meyer and D. Thong have established, using computer methods, that there are no four-valued matrices which satisfy the theorem of $COIA$ but reject &-Commutation.

Lewis quite evidently regarded &-Commutation as a correct principle; it is the first axiom of the "incontrovertible postulates" of the System of Strict Implication – not all the axioms were regarded as incontrovertible, in particular the subsequently rejected $\sim \Diamond p \to \sim \Diamond q \to . \sim p \to \sim q$ was not – and it is likewise the first axiom of all Lewis's subsequent formulations of weaker systems of strict implication (cf. [4]), p. 324 and p. 493). But if there was any real doubt the case can be clinched by extracting principles used in the "independent" proof of the paradox $p \;\&\; \sim p \to q$ (e.g. [4], p. 250). In the initial steps of the proof the Simplification principles $p \;\&\; q \to q$ and $p \;\&\; q \to p$ (both axioms of various formulations of strict implication) are applied, and in the step which ensures the conjoined antecedent of Disjunctive Syllogism, $\sim p \;\&\; (p \vee q) \to q$, the rule of Composition $A \to B$, $A \to C \to A \to (B \;\&\; C)$ is (implicitly) applied. Now simply apply Rule

Composition to the Simplification premisses given: then $p \ \& \ q \to q \ \& \ p$ results. There is also other evidence for symmetry of conjunction, e.g. it is readily derived from the first of 'the fundamental properties of disjunction', symmetry ([4], p. 135). The surprising omission of &-Commutation from the postulated of the amended *COI* can perhaps be ascribed to haste in the production of [2]. In any event its omission appears to be an oversight, and there appears to be a good for amending *COI* yet again. Let us call the finally amended system, which weakens L to L' and adds axiom

$M.$ $\quad p \ \& \ q \to q \ \& \ p$

COD. That is, *COD* is *COIA* together with M. Of course if the conjecture is wrong *COD* just is *COIA*.

The twice amended system of ordinary inference *COD* is however no partial system of $S3$; it is $S3$ itself differently formulated. A proper logic of ordinary inference would have rejected not just L, but also L', which amounts to the principle $(A \supset B \to .C \supset A) \to .A \to B \to .C \to D$ which fallaciously converts material implications into implications.

Theorem 1. *A is a theorem of COD iff A is a theorem of* $S3$*, for every wff A.*

Lemma 1. *Every theorem of COD is a theorem of* $S3$.

Proof. As the rules of *COD* are those of $S3$, it suffices to show that every axiom of *COD* is a theorem of $S3$. But all the axioms of *COD*, except those that involve $\circ$, are theorems of $S3$, as [3] shows.

ad K. Since $p \to q \to . \sim (p \ \& \sim q)$ (see [3], 37.2 and 30.31), $p \ \& \ q \to\sim (p \to\sim q)$, by Substitution, Replacement, Contraposition and Double Negation (see [3], p. 50). K then follows by definition. (Note that K is a principle that will distinguish the Sn systems from the Sn° systems of [3]; $n = 1, \ldots, 4$).

ad L'. $p \ \& \ q \supset r \ \& \ s \to . \sim (r \ \& \sim\sim s) \supset\sim (p \ \& \sim\sim q)$, by 34.1 and 32.01 (Theorem labels are those of [3]; rules used are generally not cited).

$p \ \& \ q \to r \ \& \ s \to . \sim (r \ \& \sim\sim s) \to\sim (p \ \& \sim\sim q)$, 51.11 and 32.01
$\to .r \to\sim s \to .p \to\sim q$, by 52.21, 51.11, 32.01
$\to . \sim (p \to\sim q) \to\sim (r \to\sim s)$, by 52.21, 52.22, 31.34
$\to p \circ q \to r \circ s$, by definition of $\circ$.

Observe that Lewis's derivation of L' in [1] cannot by directly adopted in view of the collapse of the strict system of [1].

LEMMA 2. *Every theorem of $S3$ is a theorem of COD.*

PROOF. Since the rules of the systems are the same, it suffices to prove that every axiom of $S3$ – as formulated, let us say, in [3] – is a theorem of COD, and that the equivalence,

$$S. \quad A \to B \leftrightarrow \sim \Diamond(A \ \& \sim B)$$

where $\Diamond A =_{Df} \sim (A \to \sim A)$, presupposed in the axiomatization of $S3$, is derivable. To establish the requisite results some systemic development of COD is inevitable.

ad Identity, $p \to p$. By G, F, and J.
ad Double Negation, $\sim\sim p \to p$, and $\sim\sim p \leftrightarrow p$. By E and Identity; and by Adjunction.
ad Contraposition, all forms. By Double Negation and Replacement.
ad Symmetry $p \ \& \ q \leftrightarrow q \ \& \ p$. By M and Adjunction.
ad $L1, \sim (r \ \& \sim s) \to \sim (p \ \& \sim q) \to .r \to s \to .p \to q$
$p \ \& \sim q \to r \ \& \sim s \to . \sim (p \to \sim\sim q) \to \sim (r \to \sim\sim s)$, from L', eliminating $\circ$.
$L1$ then results by Contraposition and J.
ad Antilogism, $p \ \& \ q \to r \to .p \ \& \sim r \to \sim q$.
$(p \ \& \sim r) \ \& \ q \to (p \ \& \ q) \ \& \sim r$, by H and Symmetry
$\sim ((p \ \& \ q) \ \& \sim r) \to \sim (p \ \& \sim r) \ \& \sim\sim q)$, by Contraposition, Double Negation.
Antilogism then follows upon applying $L1$.
ad Tautology $p \ \& \ p \leftrightarrow p$. By F and G.
ad $p \ \& \sim q \to \sim (p \ \& \sim q) \to .p \to q$
$p \ \& \sim q \to \sim (p \ \& \sim q) \to .p \ \& \ (p \ \& \sim q) \to q$, by Antilogism, Double Negation,
$\to .(p \ \& \ p) \ \& \sim q$, by H, Symmetry and Replacement
$\to .p \ \& \sim q \to q$, by Tautology and Replacement
$\to . \sim q \ \& \sim q \to \sim p$, by Symmetry, Antilogism
$\to . \sim q \to \sim p$, by Tautology, Replacement
$\to .p \to q$, by Contraposition, Replacement.

Note that some of the Replacement steps can be supplanted by use of J.

ad $p \to q \to .p \ \& \sim q \to\sim (p \ \& \sim q)$

$(p \ \& \sim q)\&(p \ \& \sim q) \to .p \ \& \sim q$, from G

$\sim (p \ \& \sim q) \to\sim ((p \ \& \sim q)\& \sim\sim (p \ \& \sim q))$, by Contraposition, Double Negation, and Replacement.

The theorem then follows by $L1$.

ad S, i.e. in unabbreviated form $A \leftrightarrow B \leftrightarrow\sim\sim (A\& \sim B) \to\sim (A\& \sim B)$. From the previous two theorems, using Contrapositions and rules.

ad $p \to \Diamond p$, i.e. $p \to . \sim (p \to\sim p)$

$p \ \& \ p \to\sim (p \to\sim p)$, by K; whence the result follows by Tautology.

ad $\circ$ Reduction, $\Diamond p \leftrightarrow .p \circ p$.

$p \to\sim p \leftrightarrow\sim \Diamond(p \ \& \sim\sim p)$, by S. Then apply Contraposition, Double Negation and Replacement.

ad $p \to q \to .\Diamond p \to \Diamond q$, i.e. $p \to q \to . \sim (p \to\sim p) \to\sim (q \to\sim q)$. $p \ \& \ p \to q \ \& \ q \to .p \circ p \to q \circ q$, by L'

$p \to q \to .\Diamond p \to \Diamond q$, by Tautology, Reduction, and Replacement.

Theorem 2. *COI is derivationally equivalent to classical two-valued logic.*

Proof. Firstly when $\to$ is represented by $\supset$, and $\circ$ by $\&$, then every theorem of COI is a classical theorem. Thus COI collapses no further than classical logic. To show it collapses all the way to classical logic it is enough, in the light of Theorem 1, to derive $\sim p \to\sim q \to . \sim \Diamond p \to\sim \Diamond q$ and to apply the result of [2], which shows that The System of Strict Implication collapses to classical logic.

ad $\sim p \to\sim q \to . \sim \Diamond p \to\sim \Diamond q$

$p \circ q \to q \circ q \to .p \ \& \ p \to q \ \& \ q$, by L

$\Diamond p \to \Diamond q \to .p \to q$, by Reduction and Tautology, whence the result follows by Contraposition.

It is a corollary of the theorems that none of the logics considered, COI, $COIA$ or COD really qualifies as a calculus of ordinary inference. For certainly such a calculus must contain $p \ \& \ q \to q \ \& \ p$, thereby ruling out $COIA$, unless it collapses to COD. But, as argued in detail in [5], neither classical two-valued logic nor strict system [5], neither classical two-valued logic nor strict system $S3$ represents a calculus of ordinary inference. The trouble with COD, which comes closest to doing the job, lies in the retention of L', a powerful but mistaken modal principles, which quickly induces paradox. For a calculus of ordinary inference one must look within the Lewis formulations, at the relevant systems of [5].

References

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Research School of Social Sciences
Australian National University