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David Miller

ON PROBABILITY MEASURES FOR REDUCTIVE SYSTEMS II

1. This paper continues the investigation of [2]. Where a, b are sentences of some formalized language, let $p(a, b)$ be a function satisfying the beautiful axiom system for probability of Popper's [3], appendix $*v$. Our problem is that of extending p to a probability-like function $q(A, B)$ that is defined for all pairs A, B of deductive systems of that language. In [2] I showed that the function q defined by the two clauses

(1) if $A \vdash B$ then $q(A, B) = \underset{b \in B}{sup}\ \underset{a \in A}{inf}\ p(a, b)$,

(2) if $A \nvdash B$ then $q(A, B) = q(AB, B)$,

was unacceptable, despite some notable successes, in that it failed to satisfy the intuitively incontestable axiom

(3) $q(A, A) = q(B, B)$

which corresponds to Popper's axiom A3. The proof (see Theorem 3 of [2]) amounts to showing that a complete unaxiomatizable system B may be considerably stronger than any of its consequences; in particular that for each sentence $x \in B$ there is a stronger sentence $y \in B$ such that $p(y, x) \leqslant 1/2$. It follows that $q(B, x) = inf\{p(y, x)/y \in B\} \leqslant 1/2$, and therefore that $q(B, B) = sup\{q(B, x)/x \in B\} \leqslant 1/2$ as well. In fact for $x \in B$ we have $q(B, x) = q(B, B) = 0$. The failure of A3 therefore seems to stem from the fact that in the definition of $q(A, B)$ the two limiting processes are performed consecutively; or anyway that it is the 'conditioned', rather than the 'conditioning', argument that is taken to the limit first. Correct or not, this diagnosis indicates that we might try some sort of double, rather than repeated, limiting process.

The example of a complete unaxiomatizable system, indeed, suggests a definition in terms of the 'limit superior' of the values of $p(a, b)$ as a and b approach A and B respectively. Although 'lim sup' is not usually defined for a partially ordered set of numbers, there is no difficulty in extending the customary definition and proving the following existence theorem.

THEOREM 1. *For any two systems A, B and for any probability measure p there is a unique number μ such that*

(i) $\forall \varepsilon > 0 \exists a \in A \exists b \in B \forall a' \in A \forall b' \in B$ *(if $a' \vdash a$ and $b' \vdash b$ then $p(a, b) < \mu + \varepsilon$),*

(ii) $\forall \varepsilon > 0 \forall a \in A \forall b \in B \exists a' \in A \exists b' \in B$ *($a' \vdash a$ and $b' \vdash b$ and $p(a, b) > \mu - \varepsilon$).*

This number μ can be written $\underset{\substack{a \to A \\ b \to B}}{\mathit{lim\ sup}}\, p(a, b)$; and we give the definition

(4) $q(A, B) = \underset{\substack{a \to A \\ b \to B}}{\mathit{lim\ sup}}\, p(a, b).$

The appearance of a genuine double limiting process in (4) is in fact illusory.

THEOREM 2. $q(A, B) = \underset{a \in A}{\mathit{inf}}\ \underset{b \to B}{\mathit{lim\ sup}}\, p(a, b).$

COROLLARY. $q(B, B) = 1.$

Thus we may just as well take for our definition of q the two clauses

(5) $q(A, B) = \underset{b \to B}{\mathit{lim\ sup}}\, p(a, b).$

(6) $q(A, B) = \underset{a \in A}{\mathit{inf}}\, q(a, B).$

If in (6) we write L for B we obtain for the absolute probability $q(A) = q(A, L)$ the identity

(7) $q(A) = \underset{a \in A}{\mathit{inf}}\, p(a),$

which was the original definition of the probability of a system suggested on p. 88 of [2].

2. We have seen already that $q(B,B)=1$, as required. And because of (7), Theorem 1 of 2 assures us that the general (unrelativized) form of the addition law

(8) $q(A)+q(B)=q(A\vee B)+q(AB)$

is satisfied. Another result easily proved is the general law of monotony. Popper's axiom B1.

THEOREM 3. $q(AB,C)\leqslant q(A,C)$.

It is the purpose of this section to show that we can also prove the special law of multiplication

(9) $q(AB)=q(A,B)\cdot q(B)$,

which holds whether or not $q(B)=0$.

THEOREM 4. $q(aB)=q(a,B)\cdot q(B)$.

PROOF. It is easily checked that $q(aB)=inf\{p(x)/x\in aB\}=inf\{p(ab)/b\in B\}$. Suppose therefore that

$$\underset{b\in B}{inf}\ p(ab)<\xi-\varepsilon<\xi+\varepsilon<\underset{b\ \to\ B}{lim\ sup}\ p(a,b)\cdot q(B).$$

(It is obvious that if the left-hand term is less then the right-hand term then suitable ξ,ε can be found). From the first inequality it is clear that there is a $b_0\in B$ such that if $B\vdash b\vdash b_0$ then $p(ab)<\xi$. From the third inequality we can deduce that $q(B)\neq 0$, so that $(\xi+\varepsilon)/q(B)limsup_{b\to B}p(a,b)$, whence $\xi/q(B)+\varepsilon>limsup_{b\to B}p(a,b)$. From the definition of $limsup$ it follows that there is a y_0 such that $B\vdash y_0\vdash b_0$ and $\xi/q(B)<p(a,y_0)$. Thus $\xi<p(a,y_0)\cdot inf\{p(b)/b\in B\}\leqslant p(a,y_0)\cdot p(y_0)$. But by the definition of b_0 and y_0 we have $p(ay_0)<\xi$, so that here there is a contradiction.

For the converse, suppose that

$$\underset{b\ \to\ B}{lim\ sup}\ p(a,b)\cdot q(B)<\xi-2\varepsilon<\xi<\underset{b\in B}{inf}\ p(ab).$$

The rightmost term is non-zero, so $q(B)$ is non-zero. Choose therefore $b_0 \in B$ such that $0 < p(b_0) < q(B) + \varepsilon$. Then $\mathit{lim\ sup}_{b \to B}\, p(a,b) \cdot (p(b_0) - \varepsilon) < \xi - 2\varepsilon$, so that $\mathit{lim\ sup}_{b \to B}\, p(a,b) \cdot p(b_0) < \xi - \varepsilon$. Since $p(b_0) \neq 0$, we have $\mathit{lim\ sup}_{b \to B}\, p(a,b) < \xi / p(b_0) - \varepsilon$. Thus there is a $b_1 \in B$ such that if $B \vdash b \vdash b_1$ then $p(a,b) < \xi / p(b_1)$. Since $b_0 b_1 \vdash b_1$ we have shown that $p(a, b_0, b_1) \cdot p(b_1) < \xi$, so that $p(a, b_0 b_1) \cdot p(b_0 b_1) < \xi$. It follows that $p(a b_0 b_1) < \xi < \mathit{inf}\{p(ab)/b \in B\}$, which is a contradiction.

Corollary. *$q(AB) = q(A, B) \cdot q(B)$.*

Proof. We simply let $a \to A$ in the theorem.

The general (relativized) addition and multiplication laws do not seem to be provable in this way, if at all.

3. A surprising consequence of the results of section 2 is contained in the following theorem.

Theorem 5. *If $q(B) \neq 0$ then $\mathit{lim}_{b \to B}\, p(a,b)$ exists.*

Proof. From Theorem 4 we know that there is a number ξ such that $\mathit{inf}_{b \in B}\ p(ab) = \xi = \mathit{lim}_{b \to B} \sup\ p(a,b) \cdot q(B)$. If the term $\mathit{lim\ sup}_{b \to B}\, p(a,b) = 0$, then clearly it is also the limit of $p(a,b)$, so that there is nothing left to prove. We shall therefore assume that $q(a, B) \neq 0$.

Choose $\varepsilon > 0$. Then since $q(B) \neq 0$ we have $\varepsilon \cdot q(B) > 0$. There is, therefore, some $b_0 \in B$ such that if $B \vdash b \vdash b_0$ then $p(ab) - \xi < \varepsilon \cdot q(B)$. Likewise there is a $b_1 \in B$ such that if $B \vdash b \vdash b_1$ then $p(b) - \xi / q(a, B) < \xi \cdot q(B)$; so that under the some conditions $q(a, B) \cdot p(b) - \xi < \varepsilon \cdot q(B)$. Consequently if $B \vdash b \vdash b_0 b_1$ then

$$|p(ab) - q(a, B) \cdot p(b)| < \varepsilon \cdot q(B).$$

Now $p(ab) = p(a,b) \cdot p(b)$ and $0 < q(B) < p(b)$, so we may divide out by $p(b)$ to obtain, for $B \vdash b \vdash b_0 b_1$,

$$|p(a,b) - q(a, B)| < \varepsilon,$$

which proves that the required limit exists.

Although $q(B) \neq 0$ is sufficient for $lim\{p(a,b)/b \to B\}$ to exist, it is surely not necessary. We have already noted that the limit exists when $\lim\sup_{b \to B} p(a,b) = 0$, and another trivial example is provided by any finitely axiomatizable system B. The situation is slightly more interesting when B is complete. A complete theory decides every a, so that $p(a,b)$ eventually settles down to either 0 or 1. And for many complete B we have $q(B) = 0$. An example is the system B constructed on pp. 93f. of [2]; the discussion in section 1 above makes it plain that for this system at any rate the double limit $lim\{(a,b)/a, b \to B\}$ fails to exist even though the repeated limit $\lim_{a \to B} \lim_{b \to B} p(a,b)$ does. It seems likely that the same may be true in many of the other cases.

4. The question obviously arises whether $lim\{p(a,b)/b \to B\}$ might exist for all systems B, whether or not $q(B) = 0$, or B is complete, or satisfies either of the other conditions noted. In the third part of this paper I shall show that if this is the case then q becomes a very well behaved measure indeed, obeying general (relativized) forms of both the addition and multiplication laws. For the moment, however, I can only offer, a conjecture, together with a hint of where a proof (or disproof) might be found.

It is known [1] that if $p(a,b)$ is a finitely additive measure on a Boolean algebra $\mathcal{A}$, and $\mathcal{A}$ is a subalgebra of $\mathcal{B}$, then there is a strictly positive measure $p^+(a,b)$ on $\mathcal{B}$ (with values in some suitable field $\mathcal{F}$ extending the reals) such that for all a and b in $\mathcal{A}$ the standard part of $p^+(a,b)$ is just $p(a,b)$. The measure p^+ can thus be thought of as an infinitesimal disturbance of p throughout the algebra $\mathcal{A}$. If for $\mathcal{A}$ we choose the Lindenbaum/Tarski algebra of the language under investigation, we can take for $\mathcal{B}$ the power set of the Stone space of $\mathcal{A}$. It follows that for any consistent system B we have $p^+(B) \neq 0$, so that $inf\{p^+(b)/b \in B\} \neq 0$, provided that this infimum exists.

However, it seems clear that the infimum will not exist in the field $\mathcal{F}$ unless $p(b)$ actually achieves its infimum in the reals. For suppose that there were a ϱ^+ in $\mathcal{F}$ such that for every $\delta \in \mathcal{F}$ there was a $b_0 \in B$ beyond which p^+ always lay within δ of ϱ^+: in symbols,

$$\exists \varrho^+ \in \mathcal{F} \forall \delta \in \mathcal{F}\ \exists b_0 \in B \forall b \text{ (if } B \vdash b \vdash b_0 \text{ then } p^+(b) - \varrho^+ < \delta).$$

If δ is infinitesimal we take standard parts, and find that if $B \vdash b \vdash b_0$ then $p(b) = \varrho$ (where ϱ is the standard part of ϱ^+). Thus for $B \vdash b \vdash b_0$ we must have $p(b) = inf\{p(b)/b \in B\}$. Similar results constrain the other Cauchy sequences involved in proving the addition and multiplication laws, and Theorem 5. It is far from obvious, therefore, that we can use such considerations to show the existence of $lim\{p^+(a,b)/b \rightarrow B\}$ even when $inf\{p^+(b)/b \in B\} \neq 0$.

The fact that $lim\{p(a,b)/b \rightarrow B\}$ exists and is actually achieved when B is complete or finitely axiomatizable is consequently, one must suppose, not enitrely without significance.

In conclusion we might note that functions such as p^+ obviously provide one sort of solution to our original problem, the problem of providing measures for deductive systems. My objection to this style of solution is not that it fails on provision, but that it fails on definition.

References

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University of Warwick
England