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A PROOF OF RYSZARD WÓJCICKI'S CONJECTURE

The note deals with some issues discussed at the Autumn School on Strongly Finite Sentential Calculi (Międzygórze 1977). The main theorem of the note was originally stated in the form of a conjecture by Ryszard Wójcicki. W. Dziobiak's example and R. Suszko's comment on the theorem are also briefly reported.

Given a sentential language $\underline{L} = (L, F_1, \ldots, F_n)$ and a structural consequence operation C on $\underline{L}$, let the symbol $Matr(C)$ denote the whole class of matrices $\mathfrak{M} = (\underline{M}, I_M)$ where $\underline{M} = (M, F_1, \ldots, F_n)$ is an algebra similar to $\underline{L}$, $I_M \subseteq M$, and $C \leqslant Cn_{\mathfrak{M}}$. In [2] it is proved that every structural consequence operation C is uniquely determined by $Matr(C)$, i.e. given two structural consequence operations C_1, C_2 on $\underline{L}$.

(1) $C_1 = C_2$ if and only if $Matr(C_1) = Matr(C_2)$.

In some special cases, the consequence C is uniquely determined also by the subclass of the class $Matr(C)$ consisting only of those C-matrices which have one-element designated set (i.e. $card(L_M) = 1$). For example, each calculus $S = (\underline{L}, C)$ implicative in the sense of Rasiowa is complete with respect to the class of so called S-algebras (cf. [1]), to be denoted here as $Alg^R(C)$. Thus, for two implicative consequence operations C_1, C_2

(2) $C_1 = C_2$ if and only if $Alg^R(C_1) = Alg^R(C_2)$,

(compare [1], [2]).

Let us turn back to the general case and assume C to be an arbitrary consequence operation. Then the symbol $Alg(C)$ will be used to denote the whole class of all C-matrices with one designated element. Elements of $Alg(C)$ will be referred to as C-*algebras*.

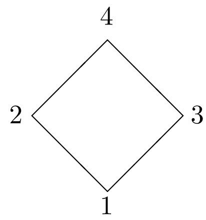
Not each structural consequence operation C can be unique determined by the class $Alg(C)$, i.e. the equivalence

(Alg) $C_1 = C_2$ if and only if $Alg(C_1) = Alg(C_2)$

is not generally true.

Example (W. Dziobiak). The modal calculus S_9 is determined by the matrix

$$\mathfrak{A} = \langle \{1, 2, 3, 4\}, \sim, \to, \Box, \{3, 4\} \rangle.$$



where $(1, 2, 3, 4, \vee, \sim)$ is the Boolean algebra illustrated by the diagram, $a \to b =^{df} \sim a \vee b$, and $\Box$ is a unary connective characterized by the following table; $\Box(1) = \Box(2) = \Box(3) =$ and $\Box(4) = 3$. One can show that $Alg(Cn_{\mathfrak{A}})$ is a one-element class and its sole element is a trivial (one-element) algebra.

Let us assume that $\mathfrak{M} = (\underline{M}, 1_{\underline{M}})$ is a non-trivial $Cn_{\mathfrak{A}}$-algebra. As in $\mathfrak{A}$ all constant functions are definable, $Cn_{\mathfrak{A}}$ is maximal and then we have $Cn_{\mathfrak{M}} = Cn_{\mathfrak{A}}$. Observe that $p \to p$, $\Box(p \to p) \in Cn_{\mathfrak{A}}(\emptyset)$ and so $(p \to p)$, $\Box(p \to p) \in Cn_{\mathfrak{M}}(\emptyset)$. Therefore $\Box 1_M = 1_M$, and consequently $\Box\Box 1_M = 1_M$. Thus $\Box\Box(p \to p) \in Cn_{\mathfrak{M}}(\emptyset)$. On the other hand, one easily verifiers that $\Box\Box(p \to p) \notin Cn_{\mathfrak{A}}(\emptyset)$. Contradiction. This implies that $Alg(Cn_{\mathfrak{A}})$ consists of the trivial algebra only. And, therefore we have $Alg(Cn_{\mathfrak{A}}) = Alg(C_L)$, where C_L denotes the inconsistent consequence operation on $\underline{L}$. But, at the same time $Cn_{\mathfrak{A}} \neq C_L$.

Given a structural consequence operation C on $\underline{L}$, $X \subseteq L$, let us put:

$$\alpha \approx_X \beta \text{ if and only if } C(X, \varphi[\alpha/p]) = C(X, \varphi[\beta/p]) \text{ for every sentential context } \varphi(p).$$

It is easy to verify that for any $X \subseteq L$, $\approx_X$ is a consequence relation on $\underline{L}$.

Theorem (R. Wójcicki's conjecture). *A structural consequence operation C on $\underline{L}$ is uniquely determined by the class $Alg(C)$ if and only if the following condition is satisfied:*

$(*)$ *for any $X \subseteq L, \alpha \in L$ such that $\alpha \in C(X)$, $C(X) = \{\beta : \beta \approx_X \alpha\}$.*

PROOF. (I) Assume that C is uniquely determined by $Alg(C)$. Let $\alpha \in C(X)$ and put:

$$Z_\alpha = \{\beta : \beta \approx_X \alpha\}.$$

(A) $C(X) \subseteq Z_\alpha$.

To prove (A) let us assume that there exists $\beta \in C(X)$ such that $\beta \notin Z_\alpha$. The second assumption implies that there is a context φ for which

$$C(X, \varphi(\alpha)) \neq C(X, \varphi(\beta)).$$

Two cases can be distinguished:

CASE 1. There is a formula α^* such that

$$\alpha^* \in C(X, \varphi(\alpha)) \text{ and } \alpha^* \notin C(X, \varphi(\beta)).$$

CASE 2. There is a formula β^* such that

$$\beta^* \notin C(X, \varphi(\alpha)) \text{ and } \beta^* \in C(X, \varphi(\beta)).$$

As the two cases are symmetric, it is enough to show a proof for only one of them. Consider for instance Case 1. Then for some algebra $\mathcal{A} = (\underline{A}, 1_A) \in Alg(C)$ and for some valuation $v : \underline{L} \to \underline{A}$

(§) $v(X, \varphi(\beta)) \subseteq \{1_A\}$ and $v(\alpha^*) \neq 1_A$.

Notice that then $v(X, \varphi(\alpha)) \not\subseteq \{1_A\}$ (if it were not the case, we would have $v(\alpha^*) = 1_A$). But that implies that $v(\varphi(\alpha)) \neq 1_A$. From (§) we also get $v(X) \subseteq \{1_A\}$. But $\alpha \in C(X)$ and $\beta \in C(X)$, so $v(\alpha) = v(\beta) = 1_A$. Consequently,

$$\begin{array}{lcl} v(\alpha) = 1_A & \text{and} & v(\varphi(\alpha)) \neq 1_A \\ v(\beta) = 1_A & \text{and} & v(\varphi(\beta)) = 1_A. \end{array}$$

This contradiction ends the proof of (A) for Case 1.

(B) $Z_\alpha \subseteq C(X)$.

Let $\beta \in Z_\alpha$. This means that $\alpha \approx_X \beta$ and we have

$$C(X, \alpha) = C(X, \beta).$$

The assumption $\alpha \in C(X)$ implies that $C(X, \alpha) = C(X)$ and therefore

$$C(X) = C(X, \beta).$$

So $\beta \in C(X)$. This ends the proof of (B) and the proof of Part (I).

(II) Assume that $(*)$ holds true. The Lindenbaum matrix

$$\mathbb{L} = \langle (\underline{L}, C(X)) : X \subseteq L \rangle$$

is strictly adequate for C, i.e. $C = Cn_{\mathbb{L}}$ (cf. [2]). Let us now consider the quotient matrix

$$\mathbb{L}/\approx = \langle (\underline{L}/\approx_X, C(X)/\approx_X) : X \subseteq L \rangle.$$

From condition $(*)$ it follows that $\approx_X$ is a matrix congruence of $(\underline{L}, C(X))$ and therefore $C = Cn_{\mathbb{L}/\approx}$. Moreover, form $(*)$ we also get that $C(X)/\approx_X$ is a one-element set. Therefore each of the quotient matrices in $\mathbb{L}/\approx$ is a C-algebra. Consequently $C = Cn_{\mathbb{L}/\approx} = inf\{Cn_{\mathcal{A}} : \mathcal{A} \in Alg(C)\}$. Hence C is uniquely determined by the class of its C algebras, $Alg(C)$. This ends the proof of the theorem. (Thanks are due to J. Zygmunt for helpful conversations on this proof.) ■

R. Suszko observed that a consequence operation C possesses property $(*)$ if and only if the rule

$$(\bullet) \quad \frac{\alpha, \beta, \varphi(\alpha)}{\varphi(\beta)} \quad (\forall\varphi)$$

is a rule of C. Thus, an alternative formulation of the theorem proved is the following:

A structural consequence operation C on $\underline{L}$ is uniquely determined by the class of C-algebras, $Alg(C)$, if and only if $(\bullet)$ is a C-rule.

References

[1] H. Rasiowa, **An algebraic approach to non-classical logics**, North Holland Publ. Co., Amsterdam, PWN, Warszawa, 1974.

[2] R. Wójcicki, *Matrix approach in methodology of sentential calculi*, **Studia Logica** 33 (1973), pp. 7–37.

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