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THE LATTICE OF RAMIFIED MODAL AND TENSE LOGIC (PRELIMINARY REPORT)

Let K^n denote the smallest n-ramified normal modal logic, $n \in \omega$. K^n is like K, except that it has a system Ω of n unary modal functors. The case $n = 2$ corresponds to a somewhat extended situation as occurs in the so-called tense logic. The (normed) matrices or models of K^n are expanded Boolean algebras with a set Ω of n unary operations, each $\blacksquare \in \Omega$ satisfying

($\circ$) $\blacksquare a \cap \blacksquare b \leqslant \blacksquare(a \cap b)$ (1) $\blacksquare 1 = 1$

This, each $\blacksquare \in \Omega$ is a *generalized kernel operator*. By Q^nBA we denote the class of these expanded Boolean algebras. Let N^n denote the complete lattice of normal extensions of K^n. Clearly, if $L \in N^n$, then $L = LA$, some $A \in Q^nBA$. If A is finite, L is said to be *tabular*.

Theorem 1. *Q^nBA is congruence-distributive (i.e. each member $A \in Q^nBA$ is). Therefore N^n is distributive.*

Moreover, we have the following

Theorem 1′. *N^n is a HEYTING-algebra.*

This follows from the Lemma below whose proof is based on a careful analysis of the formal definition on the $L \in N^n$. By $L(P)_{P \in X}$ we denote the logic in N^n determined by the axiom system X. $ⓜP$ is the conjunction of all ΠP, Π runs over all prefixes $\blacksquare_1 \ldots \blacksquare_l$ of length l, $0 \leqslant l \leqslant m$.

Lemma. $L(P)_{P \in X} \cap L(Q)_{Q \in Y} = L(ⓜP \vee ⓜQ)_{P \in X; Q\ Y; m \in \omega}$.

Theorem 2. *The tabular $L \in N^n$ form a sublattice of N^n.*

Theorem 3. *Each tabular $L \in N^n$ has finitely many extensions only.*

Conversely, if $L \in N (:= N^1)$ has finitely many extensions only, then L is tabular.

Theorem 3 is not true $n \geqslant 2$, as follows from an incompleteness result of S. K. Thomason.

$L \in N^n$ is said to be *pretabular*, if L is not tabular, but each proper extension of L is.

THEOREM 4. *Each non-tabular $L \in N^n$ is contained in some pretabular $L \in N^n$.*

The Theorems 3 and 4 are essentially based on the following criterion. Put $\chi_k^n := \bigvee_{0 \leqslant i \leqslant j \leqslant n} ⓚ(p_i \leftrightarrow p_j) \bigwedge_{..} ⓚp_0 \rightarrow \boxed{k+1}p_0$.

CRITERION. *$L \in N^n$ is tabular <u>iff</u> $\chi_k^n \in L$ for some $k \in \omega$.*

THEOREM 5. *The pretabular $L \in N$ have the form $L = \bigcap_{i \in \omega} L_i$, each L_i prime tabular and $L_0 \supset L_1 \supset L_2 \supset \ldots$.*

$L \in N^n$ is said to be *locally finite*, if each finitely generated $A \in ModL$ is finite. K_m^n denotes the set of extensions of $K^n(ⓜp \rightarrow \boxed{m+1}p)$.

THEOREM 6. *If $L \in K_m^n$ is locally finite, then $ModL$ contains an infinite subdirect irreducible $A \in Q^n BA$ iff $ModL$ contains an infinite number of finite subdirect irreducible members.*

Theorem 6 is based on the following

CRITERION. *$A \in Q^n BA$ is subdirect irreducible <u>iff</u> $\exists c \in A : \forall a \in A : \exists n \in \omega : ⓝa \leqslant c$.*

Let P_n be the formula expressing "In Kripke-frames there are path's of length at most n".

THEOREM 7. *If $P_m \in L \in N^n$ then L is locally finite.*

COROLLARY. *$L \in N_m^n$ is tabular <u>iff</u> L has finitely many extensions only.*

From the Lemma above it follows also that the finitely axiomatizable $L \in N_m^n$ form a sublattice of N_m^n; it is doubtful whether this is true for $N = N^1$.

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