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A THEOREM ON VERSIMILITUDE

This paper is a shortened version of one read to the Philosophy Seminar, Australian National University, October 1977. The main result, and additional results, are being prepared for publication in the author's 'Relevance and Verisimilitude'.

Recent work on Proper's qualitative theory of verisimilitude by Miller ([3], [4], [5]), Tichy ([13], [14]) and Propper ([7]) has indicated that there is considerable difficulty in giving a satisfactory account of the idea of one theory's being nearer to the truth than another. On the other hand, intuition seems to support the existence of such a notion (though this cannot be regarded as a decisive reason). Moreover, Putnam ([9]) and Smart ([12]) have both argued that scientific realism requires a notion of nearness to the truth (see also Popper [8], pp. 223 ff.)

Popper's (original) qualitative theory of verisimilitude is taken here to be following

Definition 1. Let T be the set of true sentences (relative to a language $\mathcal{L}$), $F = \overline{T}$ (relative to L) and A and B be theories whose language is $\mathcal{L}$. Then $B >_v A$ (B *is more verisimilar than* A iff either $A \cap T \subseteq B \cap T$ and $B \cap F \subset A \cap F$, or $A \cap T \subset B \cap T$ and $B \cap F \subseteq A \cap F$.

We will also need the following definitions

Definition 2. Let L be a logic with a suitable implication operation '$\rightarrow$' and a suitable conjunction operation '&'. A set A is an *L-theory* iff (1) if $\alpha \in A$ and $\vdash_L \alpha \rightarrow \beta$, then $\beta \in A$, and (2) if $\alpha \in A$ and $\beta \in A$, then $\alpha \ \& \ \beta \in A$. (Sometimes, as in classical logic, the second condition can be dropped.)

Definition 3. Let A be an L-theory. *Then the rule* γ holds for A iff, if $\alpha \in A$ and $\sim \alpha \vee \beta \in A$, then $\beta \in A$.

Miller and Tichy's result is set out below, together with remarks on the proof, which are intended to draw attention to principles used in the proof. The proof is close to that of Tichy ([13]), and Harris [(2]).

Theorem 1. (Miller-Tichy) *If A and B are theories and $B >_v A$, then $B \cap F = \Lambda$.*

Proof. If $B >_v A$, then either (1) $A \cap T \subset B \cap T$ and $B \cap F \subseteq A \cap F$, or (2) $A \cap T \subseteq B \cap T$ and $B \cap F \subset A \cap F$. On either supposition, together with the assumption that $B \cap F \neq \Lambda$ we can deduce a contradiction.

(1) Suppose $A \cap T \subset B \cap T$ and $B \cap F \subseteq A \cap F$ and, for contradiction, suppose $B \cap F \neq \Lambda$.

Let $f \in B \cap F$. Let $b \in B \cap T - A \cap T$. Since $f \in B \cap F$, $f \in B$. Since $b \in B \cap T$, $b \in B$. Hence $f \,\&\, b \in B$ (Remark (a)). Also, since $f \in B \cap F$, $f \in F$. Hence $f \,\&\, b \in F$ (Remark (b)). So $f \,\&\, b \in B \cap F$. We show that $f \,\&\, b \notin A \cap F$, thereby contradicting the assumption that $B \cap F \subseteq A \cap F$. (For suppose $f \,\&\, b \in A \cap F$. Then $f \,\&\, b \in A$, so $b \in A$ (Remark (c)). But since $b \in B \cap T$, $b \in T$, so $b \in A \cap T$, contradicting the assumption that $b \notin A \cap T$. Hence $f \,\&\, b \notin A \cap F$)

(2) Suppose $A \cap T \subseteq B \cap T$ and $B \cap F \subset A \cap F$ and, for contradiction, suppose $B \cap F \neq \Lambda$.

Let $f \in B \cap F$. Let $a \in A \cap F - B \cap F$. Since $f \in B \cap F$, $f \in F$. So $\sim f \in T$. (Remark (d)). So $\sim f \vee a \in T$ (Remark (e)). Since $a \in A \cap F$, $a \in A$, so $\sim f \vee a \in A$. (Remark (f)). Hence $\sim f \vee a \in B \cap T$. But $A \cap T \subseteq B \cap T$, so $\sim f \vee a \in B \cap T$. So $\sim f \vee a \in B$. But $f \in B \cap F$, so $f \in B$. *From $\sim f \vee a \in B$ and $f \in B$, we may deduce $a \in B$.* (Remark (g)). But $a \in A \cap F$, so $a \in F$, so $a \in B \cap F$, contradicting the assumption that $a \in A \cap F - B \cap F$.

Remarks (a) From $f \in B$ and $b \in B$ to deduce $f \,\&\, b \in B$. This is part of the definition of an L-theory.

(b) From $f \in F$ to deduce $f \,\&\, b \in F$. This is a natural, though perhaps not inevitable, restriction to place on the complement of the set of truths (=, perhaps, the set of falsehoods)

(c) From $f \mathbin{\&} b \in A$ to deduce $b \in A$. If A is an L-theory and $\vdash_L (X \mathbin{\&} Y) \rightarrow Y$, then this move is correct. Moreover, $\vdash_L (X \mathbin{\&} Y) \rightarrow Y$ is a natural theorem to adopt, thought it is disputed in connexive logic.

(d) From $f \in F$ to deduce $\sim f \in T$. This move amounts to the assumption of completeness for T (Either $X \in T$ or $\sim X \in T$, for any X). The assumption is natural, though of course debatable if T contains arithmetic.

(e) From $\sim f \in T$ to deduce $\sim f \vee a \in T$. This is a move like (b) above. It is a natural condition on T.

(f) From $a \in A$ to deduce $\sim f \vee a \in A$. This is a move like (c) above. It is permissible provided that A is an L-theory where $\vdash_L X \rightarrow (Y \vee X)$, and this is a natural theorem to adopt, though disputed in conceptivist logic.

(g) From $\sim f \vee a \in B$ and $f \in B$ to deduce $a \in B$. This is the assumption that γ holds for the (arbitrarily chosen) theory B.

The important point for our purposes here is that the proof assumes that all theories satisfy the rule γ. All theories of classical logic satisfy γ, and of course it is classical logic which is assumed by Miller and Tichy. Thus the correct statement of Theorem 1 is 'If A and B are classical theories, then ...'. As is easy to see, the assumption that γ holds for a given theory is sufficient to ensure that if the theory is (negation) inconsistent, it is trivial.

However, γ fails for large classes of theories based on logics other than classical logic. Various reasons have been given for rejecting γ as a universally valid rule of inference (see e.g. Anderson and Belnap [1], Routley and Meyer [11], and Routley [10]). These reasons, needless to say, go along with a rejection of classical logic as the correct logic for large classes of ordinary inferential situations, such as those in mathematics or science. It is not, of course, enough to point out that various of the principles used in the proof of Theorem 1 can be made to fail in some logic or other. In order to weaken the significance of Theorem 1, it needs to be argued that some of those principles must be dispensed with in a logic adequate for mathematics or science. However, strong candidates for such a logic are to be found among relevant logics. In this connection one argument (among the many) seems persuasive. Scientific theorising typically proceeds in ignorance of the consistency is uncovered, and since inconsistencies are false (a stronger position might hold that inconsistencies are merely usually false) they need to be removed. The means of removing them might not be immediately obvious, however, and so research continues in the context of a known false (because known inconsistent) theory. But this activity is unintelligible if the background logic is classical (indeed, intuitionist): any theorem we wish to deduce from our theory can be deduced immediately from the contradiction.

Might Theorem 1 be reproved so as to apply to relevant logics as well as classical logics? The following theorem shows not.

THEOREM 2. *There exist two $RM3$-theories A, B, such that $B >_v A$ and $B \cap F \neq \Lambda$.*

PROOF. Let the language $\mathcal{L}$ be determined by a denumerable number of constants $\{p_1, p_2, \ldots\}$ closed under negations and conjunctions. The theories we construct will thus be zero degree theories, containing no occurrences of '$\to$'. Clearly, however, the result applies to theories of higher degree, indeed of higher order. We need the $RM3$-matrices (on $RM3$, see [1]).

&	T	N	F	$\sim$
$*T$	T	N	F	F
$*N$	N	N	F	N
F	F	F	F	T

$\to$	T	N	F
$*T$	T	F	F
$*N$	T	N	F
F	T	T	T

A is constructed as follows

(1) For all $n \geqslant 0$, $V_A(\sim^n p_1) = V_A(\sim^n p_2) = N$
(2) For all $n \geqslant 0$ and $m \geqslant 3$, $V_A(\sim^{2n} p_m) = T$ and $V_A(\sim^{2n+1} p_m) = F$
(3) If α is of the form β & γ, then $V_A(\alpha)$ is determined from the $RM3$-table for '&'.
(4) If α is of the form $\sim^n (\beta \mathbin{\&} \gamma)$ for some $n \geqslant 1$, then $V_A(\alpha)$ is determined from the $RM3$-table for '$\sim$'.

And let $A = \{\alpha : V_A(\alpha) = T \text{ or } V_A(\alpha) = N\}$.

B is constructed as follows

(1′) For all $n \geqslant 0$, $V_B(\sim^n p_1) = N$
(2′) As for (2) with '$m \geqslant 3$' replaced by '$m \geqslant 2$'.
(3′) = (3)
(4′) = (4)

And Let $B = \{\alpha : V_B(\alpha) = T \text{ or } V_B(\alpha) = N\}$

Finally we construct a theory B^- which will be proved can be taken for T.

(1") As for (2) above, with '$m \geqslant 3$' replaced by '$m \geqslant 1$'
(2") $= (3)$
(3") $= (4)$

And let $B^- = \{\alpha : V_{B^-}(\alpha) = T \text{ or } V_{B^-}(\alpha) = N\}$.

The theorem then follows from the following five lemmata

LEMMA 1. *A, B, B^- are RM3-theories.*

LEMMA 2. *$B^- \subset B \subset A$.*

LEMMA 3. *A is nontrivial.*

LEMMA 4. *B^- is a classical theory, and is negation consistent and complete in $\mathcal{L}$.*

LEMMA 5. *Let $T = B^-$. Then $A \cap T = B \cap T = T$ and $B \cap F \neq \Lambda$.*

The importance of the selection of the selection of $RM3$ as a background logic is that a large class of plausible relevant logics are weaker than $RM3$, which immediately enables us to deduce

THEOREM 3. *There are two T-, E-, R-, EM-, RM-, …, etc., theories such that $B >_v A$ and $B \cap F \neq \Lambda$. (On T, etc., see [1]).*

We conclude with some more speculative remarks. As noted earlier, intuition seems to support the viability of a notion of verisimilitude. In addition, intuition suggests that Popper's definition provides at least a sufficient condition for one theory's being closer to the truth than another. The import of the Miller-Tichy result is that not very many pairs of classical theories satisfy this sufficient condition. But relevant mathematics can still avail itself of that sufficient condition, and perhaps even take it for a necessary condition as well. In connection with this point, it is somewhat less reasonable to demand that an account of verisimilitude impose an interesting ordering on *all* theories, than it is to demand that the account of verisimilitude yield interesting results for theories which are either inconsistent or incomplete.

The results given here do not *by themselves* solve the 'problem of verisimilitude', even for relevant logics. The problem of finding a way out of the Miller-Tichy result for those theories for which γ holds remains. Research into this problem is continuing, particularly in connection with escaping the first part of Theorem 1, which does not depend on γ. Conceivably, however, it might be reasonable to conclude that Popper's definition of verisimilitude is the best we can come up with, and that relevant mathematics can be equipped with a satisfactory account of verisimilitude which does not, however, impose an interesting ordering on those relevant theories which are also classical. In such a situation, classical logic would at a clear disadvantage with respect to relevant logic.

References

[1] A. R. Anderson, and N. D. Belnap, **Entailment**, Volume One, Princeton University Press, 1975.

[2] J. Harris, *Popper's Definition of Verisimilitude*, **British Journal for the Philosophy of Science**, Vol. 25 (1974), pp. 160–166.

[3] D. Miller, *Popper's Qualitative Theory of Verisimilitude*, **British Journal for the Philosophy of Science**, Vol. 25 (1974), pp. 166–177.

[4] D. Miller, *On the Comparison of False Theories by their Bases*, **British Journal for the Philosophy of Science**, Vol. 25 (1974), pp. 178–188.

[5] D. Miller, *Verisimilitude Redeflated*, **British Journal for the Philosophy of Science**, Vol. 27 (1976), pp. 363–381.

[6] D. Miller, *On Distance from the Truth as a True Distance*, **Bulletin of the Section of Logic**, Vol. 6 no. 1 (March 1977), pp. 15–26.

[7] K. Popper, *A Note on Verisimilitude*, **British Journal for the Philosophy of Science**, Vol. 27 (1976), pp. 147–159.

[8] K. Popper, **Conjectures and Refutations**, London, Routledge and Kegan Paul, 1963.

[9] H. Putnam, *What is 'Realism'?*, **Proceedings of the Aristitelian Society**, Vol. 76 (1975-6), pp. 177–194.

[10] R. Routley, *Ultralogic as Universal?*, **Relevance Logic Newsletter**, Vol. (Jan. and May 1977), pp. 50–95 and 138–175.

[11] R. Routley and R. Meyer, **Relevant Logics and Their Rivals**, Research School of Social Sciences, Australian National University, 1977.

[12] J. J. C. Smart, **Science as an Approximation to the Truth**, Paper read to A.A.H.P.S. Conference, Melbourne, August 1976.

[13] P. Tichy, *On Propper's Definitions of Verisimilitude*, **British Journal for the Philosophy of Science**, Vol. 25 (1974), pp. 155-160.

[14] P. Tichy, *Verisimilitude Redefined*, **British Journal for the Philosophy of Science**, Vol. 27 (1976), pp. 25–42.

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