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DUMMETT'S LOGIC HAS THE INTERPOLATION PROPERTY

This is an abstract of one part of the paper which will appear in Reports on Mathematical Logic.

The logic LC is obtained by adding the axiom schema $(\alpha \to \beta) \vee (\beta \to \alpha)$ to the intuitionistic propositional logic. We will prove that LC has the Interpolation Property, i.e. if $\alpha \to \beta \in LC$ and $Var(\alpha) \cap Var(\beta) \neq \emptyset$, then there exists a formula γ in which only the variables from $Var(\alpha) \cap Var(\beta)$ occur, such that $\alpha \to \gamma$, $\gamma \to \beta \in LC$ ($Var(\alpha)$ denotes the set of variables occurring in α).

Let Γ be a finite set of propositional variables and let $FOR(\Gamma)$ denote the set of all formulas built up from the variables from Γ. It is well known that the quotient of $FOR(\Gamma)$ modulo LC-provable equivalence is finite. Let us denote by $F(\Gamma)$ an arbitrary selector of this finite set.

It is also well known that LC is complete with respect to finite linear Kripke models (called models in the sequel), i.e. $\alpha \in LC$ iff $a \Vdash \alpha$ for every model $\mathcal{A} = \langle A, \leqslant, \Vdash \rangle$ and every $a \in A$.

For every model $\mathcal{A} = \langle A, \leqslant, \Vdash \rangle$ and every $a \in A$ let us define:

$$\begin{aligned} a(\Gamma, \mathcal{A}) &= \{x \in \Gamma : a \Vdash x\}, \\ a^*(\Gamma, \mathcal{A}) &= \{b(\Gamma, \mathcal{A}) : b \geqslant a\}, \\ [a]\,(\Gamma, \mathcal{A}) &= \bigwedge\!\!\!\bigwedge \{\alpha \in F(\Gamma) : a \Vdash \alpha\}. \end{aligned}$$

Lemma. *If $\mathcal{A} = \langle A, \leqslant, \Vdash \rangle$, $\mathcal{B} = \langle B, \leqslant, \Vdash \rangle$ are models, $a \in A$, $b \in B$ and $b \Vdash [a]\ (\Gamma, \mathcal{A})$, then there exists $c \in A$, $c \geqslant a$ such that $b^*(\Gamma, \mathcal{B}) = c^*(\Gamma, \mathcal{A})$.*

Theorem. *LC has the Interpolation Property.*

Sketch of the proof. Let $\alpha \to \beta \in LC$ and let $\Gamma = Var(\alpha) \cap Var(\beta)$ be nonempty. Define:

$$\gamma = \bigvee\{[a](\Gamma, \mathcal{A}) : \mathcal{A} = \langle A, \leqslant, \Vdash\rangle \text{ is a model, } a \in A, a \Vdash \alpha\}.$$

We will prove that γ is the interpolant of $\alpha \to \beta$.

1. If $\mathcal{A} = \langle A, \leqslant, \Vdash\rangle$ is a model, $a \in A$ and $a \Vdash \alpha$, then $[a](\Gamma, \mathcal{A})$ is one of the disjuncts of γ. Obviously, $a \Vdash [a](\Gamma, \mathcal{A})$ and hence $a \Vdash \gamma$.

2. Suppose $\gamma \to \beta \notin LC$. Then there exist a model $\mathcal{B} = \langle B, \leqslant, \Vdash\rangle$ and $b \in B$ such that $b \Vdash \gamma$, $b \nVdash \beta$. From the definition of γ there exists a model $\mathcal{A} = \langle , \leqslant, \Vdash\rangle$ and $a \in A$ such that $b \Vdash [a](\Gamma, \mathcal{A})$. From the lemma $b^*(\Gamma, \mathcal{B}) = c^*(\Gamma, \mathcal{A})$ for some $c \in A$, $c \geqslant a$. One can establish the existence of a finite linearly ordered set $\langle C, \leqslant\rangle$ and mappings f, g satisfying the following conditions:

a) $f : C \to \{d \in A : d \geqslant a\}$,
   $g : C \to \{d \in B : d \geqslant b\}$,

b) f, g are onto and monotone,

c) for every $c \in C : f(c)(\Gamma, \mathcal{A}) = g(c)(\Gamma, \mathcal{B})$.

f, g are strong homomorphisms onto their ranges (see [1]). Let us define now the forcing relation $\Vdash$ on $\langle C, \leqslant\rangle$ putting for every $d \in C$:

if $x \in Var(\alpha)$, then $d \Vdash x$ iff $f(d) \Vdash x$,

if $x \in Var(\beta)$, then $d \Vdash x$ iff $g(d) \Vdash x$,

and arbitrary for remaining variables. Condition c) guarantees that the definition is consistent. It follows from De Jongh's theorem on strong homomorphisms (see [1]) that for every $\delta \in FOR(Var(\alpha))$ $d \Vdash \delta$ iff $f(d) \Vdash \delta$ and for every $\delta \in FOR(Var(\beta))$ $d \Vdash \delta$ iff $g(d) \Vdash \delta$. Consider now the model $\langle C, \leqslant, \Vdash\rangle$ and $d \in C$ such that $g(d) = b$. Obviously, $d \nVdash \beta$ (since $b \nVdash \beta$). On the other hand, $d \Vdash \alpha$ (since $a \Vdash \alpha$ and $f(d) \geqslant a$). It is a contradiction with $\alpha \to \beta \in LC$.

References

[1] C. Smorynski, **Investigations of the intuitionistic formal systems by means of Kripke models**, Ph. D. Thesis, Stanford University, 1972.

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